MATH 700
HOMEWORK 6

Due Friday, October 17, 2003 at the beginning of class.

The vector spaces in the assignment have arbitrary dimension. All direct sums are internal.

1. **Let V_1 and V_2 be subspaces of the vector space V. Suppose that $V_1 + V_2 = V$. Prove that there exists a subspace W of V such that $W \subseteq V_2$ and $V = V_1 \oplus W$.**

If $V_1 = V$, then take $W = \{0\}$. Henceforth, we assume $V_1 \neq V$. Let S_1 be a basis for V_1 and S_2 be a basis for V_2. I will find T, a subset of S_2, such that $S_1 \cup T$ is a basis for V. At that point, I let W be the span of T, and I am finished. I use Zorn's Lemma to prove that T exists. Let

$$\mathcal{P} = \{S \subseteq S_2 \mid S_1 \cup S \text{ is linearly independent}\}.$$

We see that $\mathcal{P}$ is non-empty, and that $\mathcal{P}$, together with $\subseteq$, is a poset. If $\mathcal{T} = \{S_a \mid a \in A\}$ is a chain in $\mathcal{P}$, then the union $U = \bigcup\limits_{a \in A} S_a$ is an upper bound for $\mathcal{T}$, which is in $\mathcal{P}$. (We know that $S_1 \cup U$ is linearly independent because any relation would involve only a finite number of S_a's and hence, would occur in $S_1 \cup S_{a_0}$, for some a_0 (since the S_a's are totally ordered!). But $S_1 \cup S_{a_0}$ is linearly independent.) Zorn's lemma tells us that $\mathcal{P}$ has a maximal element T. It is clear that $S_1 \cup T$ is linearly independent. If $S_1 \cup T$ did not span V, then there would exist v in S_2 such that $v \notin T$ and $S_1 \cup T \cup \{v\}$ is linearly independent. In this case, $T \cup \{v\}$ would be an element of $\mathcal{P}$ which is properly larger than the maximal element T of $\mathcal{P}$. This is impossible. We conclude that $S_1 \cup T$ is a basis for V.

2. **(Answer each question with a COMPLETE proof or a counterexample.) Let X, Y, and Z be subspaces of the vector space V with $X \oplus Y = V$ and $X \oplus Z = V$. Is $Y = Z$? Is $Y \cong Z$?**

It is easy to see that Y does not have to equal Z. Let $V = \mathbb{R}^2$, X be the space spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, Y be the space spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and Z be the space spanned by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. It is clear that $X \oplus Y = V = X \oplus Z$ and $Y \neq Z$. On the other hand, Y is isomorphic to Z. Define $\varphi\colon Y \to Z$ as follows. Each element of V can be written in the form $x + z$, for some $x \in X$ and some $z \in Z$, in a unique manner! For $y \in Y$ define $\varphi(y) = z$ where z is the unique element of Z with $y - z \in X$. It is not difficult to see that $\varphi\colon Y \to Z$ is a homomorphism. Take y and y' from Y and $c \in F$, where F is the field of scalars. Let z and z' be the unique elements of Z with $y - z \in X$ and $y' - z' \in X$. We see that $(cy + y') - (cz + z') = c(y - z) + (y' - z') \in X$; hence, $\varphi(cy + y') = cz + z' = c\varphi(y) + \varphi(y')$. We show that φ is one-to-one. If $y \in Y$ and $\varphi(y) = 0$, then $y \in Y \cap X = \{0\}$. We show that φ is onto. Take $z \in Z$. We know that $z \in V = X \oplus Y$. So, $z = x + y$ for some $x \in X$ and $y \in Y$. Observe that $y - z = x \in X$; hence, $\varphi(y) = z$.

3. **Suppose that $T\colon V \to V$ is a linear transformation on the vector space V which satisfies $TT = T$. Prove that $V = \ker T \oplus \operatorname{im} T$.**

If $v \in V$, then $v = [v - T(v)] + T(v)$. We see that $v - T(v) \in \ker T$ and $T(v) \in \operatorname{im} T$. Thus, $V = \ker T + \operatorname{im} T$. If $v \in \ker T \cap \operatorname{im} T$, then $v = T(v')$ for some $v' \in V$ and $0 = T(v) = TT(v') = T(v') = v$.