MATH 700
HOMEWORK 5

Due Friday, October 10, 2003 at the beginning of class.

Definition. If V is a vector space over the field F, then the *dual of* V is the vector space $V^* = \mathrm{Hom}_F(V, F)$. If V and W are vector spaces over the field F, and $T\colon V \to W$ is a linear transformation, then the *dual of* T is the transformation $T^*\colon W^* \to V^*$ which is defined by $T^*(\varphi) = \varphi \circ T$ for all $\varphi \in W^*$.

Definition. If $A = (a_{ij})$ is an $n \times n$ matrix with entries in the field F, then the *trace* of A is $\sum_{i=1}^n a_{ii}$.

1. **(Hoffman and Kunze page 116, number 6) Let V be the vector space of polynomials of degree at most n over $\mathbb{R}$. Let $D\colon V \to V$ be differentiation. Find a basis for the null space of D^*.**

The most convenient basis for V is $1, x, x^2, \dots, x^n$. The dual basis for V^* is $\varphi_0, \varphi_1, \dots, \varphi_n$, where $\varphi_i(x^j) = \delta_{i,j}$ for $0 \le i, j \le n$. We know that $D^*(\varphi_i) = \varphi_i \circ D$. If $D^*(\varphi_i)$ is applied to x^j, then the answer is $\varphi_i(jx^{j-1}) = j\delta_{i,j-1} = (i+1)\delta_{i+1,j}$. Of course, if $0 \le i \le n-1$, then $(i+1)\varphi_{i+1}$ is the element of V^* which sends each x^j from V to $(i+1)\delta_{i+1,j}$. We conclude that

$$D^*(\varphi_i) = \begin{cases} (i+1)\varphi_{i+1} & \text{if } 0 \le i \le n-1 \\ 0 & \text{if } i = n. \end{cases}$$

In other words, $D^*(\sum_{i=0}^{n} c_i\varphi_i) = \sum_{i=0}^{n-1}(i+1)c_i\varphi_{i+1}$, which is zero if and only if $c_0 = c_1 = \cdots = c_{n-1} = 0$. We conclude that $\boxed{\varphi_n \text{ is a basis for the null space of } D^*.}$

2. **(Hoffman and Kunze page 116, number 7) Let V be a finite dimensional vector space over the field F. Let $\varphi\colon \mathrm{Hom}_F(V, V) \to \mathrm{Hom}_F(V^*, V^*)$ be the function which is defined by $\varphi(T) = T^*$ for all $T \in \mathrm{Hom}_F(V, V)$. Prove that φ is an isomorphism of vector spaces.**

It is clear that φ is a homomorphism. Take $S, T \in \mathrm{Hom}_F(V, V)$. We know that $\varphi(S + T) = (S + T)^*$ and $\varphi(S) + \varphi(T) = S^* + T^*$. Observe that $(S + T)^*$ and $S^* + T^*$ are the same element of $\mathrm{Hom}_F(V^*, V^*)$ because if $\alpha \in V^*$, then $(S + T)^*$ takes α to $\alpha \circ (S + T)$ and $S^* + T^*$ takes α to $S^*(\alpha) + T^*(\alpha) = \alpha \circ S + \alpha \circ T = \alpha \circ (S + T)$. In a similar manner, we see that $\varphi(cS) = (cS)^*$ and $c\varphi(S) = c(S^*)$ for any c in F. The elements $(cS)^*$ and $c(S^*)$ of $\mathrm{Hom}_F(V^*, V^*)$ are equal because if $\alpha \in V^*$, then $(cS)^*$ takes α to $\alpha \circ (cS)$ and $c(S^*)$ takes α to $c(\alpha \circ S)$. These two elements of V^* do the same thing to each element v of V.

Now we show that the homomorphism φ is one-to-one. Suppose that T is in the null space of φ. Then $T^*\colon V^* \to V^*$ is the zero homomorphism. Thus,

$\alpha \circ T\colon V \to F$ is the zero homomorphism for all $\alpha \in V^*$. If $v \in V$, then $\alpha(T(v)) = 0$ for all $\alpha \in V^*$. This tells me that $T(v)$ must be zero. (If $T(v)$ is not zero, then it is part of a basis for V and I can find α in V^* which keeps $T(v)$ alive.) So, $T(v)$ is zero for all $v \in V$. Thus, T is the zero linear transformation.

The vector spaces $\operatorname{Hom}_F(V,V)$ and $\operatorname{Hom}_F(V^*,V^*)$ have the same dimension over F, namely $(\dim V)^2$. The dimension of the image of φ plus the dimension of the null space of φ is equal to the dimension of $\operatorname{Hom}_F(V,V)$. It follows that the dimension of the image of φ is equal to the dimension of $\operatorname{Hom}_F(V^*,V^*)$ and φ is onto.

3. **(Hoffman and Kunze page 116, number 8) Let V be the vector space of $n \times n$ matrices over the field F.**
   - **(a) If $B \in V$, then define the function $f_B\colon V \to F$ by $f_B(A) = \operatorname{trace}(B^{\mathrm{T}}A)$. Prove that f_B is a linear transformation. (In this problem B^{T} is the transpose of the matrix B.)**
   - **(b) Let $\Phi\colon V \to V^*$ be the function defined by $\Phi(B) = f_B$ for all $B \in V$. Prove that Φ is onto.**
   - **(c) Prove that the function Φ of (b) is an isomorphism.**

For (a), the matrix B is fixed. We are to study $f_B\colon V \to F$. Take A and C in V. (So, A and C are matrices.) Observe that

$$f_B(A+C) = \operatorname{trace}(B^{\mathrm{T}}(A+C)) = \operatorname{trace}(B^{\mathrm{T}}A + B^{\mathrm{T}}C)$$

(because matrix multiplication distributes)

$$= \operatorname{trace}(B^{\mathrm{T}}A) + \operatorname{trace}(B^{\mathrm{T}}C)$$

(just look at the definition of trace, the trace of a sum is the sum of the traces)

$$= f_B(A) + f_B(C).$$

Take $c \in F$. Observe that

$$f_B(cA) = \operatorname{trace}(B^{\mathrm{T}}cA) = \operatorname{trace}(cB^{\mathrm{T}}A)$$

(because I can stick the constant wherever I want when I multiply the matrices)

$$= c\operatorname{trace}(B^{\mathrm{T}}A)$$

(again look at the definition of trace: the trace of a constant times a matrix is the constant times the trace of the matrix)

$$= cf_B(A).$$

For (b), my favorite basis for V is $\{E_{i,j}\}$, where $E_{i,j}$ is the $n \times n$ matrix with 1 in position (i,j) and zero everywhere else. The dual basis for V^* is $\{e_{i,j}\}$, where $e_{i,j}\colon V \to F$ is the linear transformation which sends $E_{k,\ell}$ to one if $i = k$

and $j = \ell$ and $e_{i,j}$ sends $E_{k,\ell}$ to zero otherwise. Observe that $\Phi(E_{i,j}) = e_{i,j}$ because $\Phi(E_{i,j}) = f_{E_{i,j}}$ and $f_{E_{i,j}}$ sends $E_{k,\ell}$ to

$$\text{trace}(E_{i,j}^{\text{T}} E_{k,\ell}) = \text{trace}(E_{j,i} E_{k,\ell}) = \text{trace}(\delta_{i,k} E_{j,\ell}) = \delta_{i,k}\delta_{j,\ell}.$$

For (c), we first check that Φ is a homomorphism. Take $c \in F$, B and C in V. We see that $\Phi(cB + C) = f_{cB+C}$ and $c\Phi(B) + \Phi(C) = cf_B + f_C$. We show that these elements of V^* are equal by showing that they do the same thing to each element of V. Let A be an element of V. We see that

$$f_{cB+C}(A) = \text{trace}((cB + C)^{\text{T}} A) = \text{trace}(cB^{\text{T}} A + C^{\text{T}} A)$$

$$= c\,\text{trace}(B^{\text{T}} A) + \text{trace}(C^{\text{T}} A) = cf_B(A) + f_C(A) = (cf_B + f_C)(A).$$

The linear transformation is one-to-one by the rank nullity theorem since we have already shown that the dimension of the image of Φ is equal to the dimension of the domain of Φ.