MATH 700
HOMEWORK 3

Due Friday, September 12, 2003 at the beginning of class

1. Let $T\colon V \to W$ and $S\colon W \to V$ be linear transformations of vector spaces. Suppose that the composition ST is the identity map on V.
   a. If V and W have the same finite dimension, then prove that T is an isomorphism.
   b. Give an example where T is not an isomorphism, but V and W are both finite dimensional.
   c. Give an example where T is not an isomorphism, but V and W have the same infinite dimension.
2. Let V be a vector space over the field F and let $T\colon V \to V$ be a linear transformation with the property that the composition TT is the identity map on V.
   a. Assume that 2 is not the zero element of F. Prove that there are subspaces M and N of V which satisfy the all of the following properties: $M + N = V$, $M \cap N = 0$, $T(\alpha) = \alpha$ for all $\alpha \in M$, and $T(\alpha) = -\alpha$ for all $\alpha \in N$.
   b. Give an example which shows that part (a) is false when F is the field with two elements. NOTE: Write your example up carefully! You must show exactly which property fails.
3. Give an example a function $T\colon \mathbb{C} \to \mathbb{C}$ such that T is a linear transformation when $\mathbb{C}$ is viewed as a vector space over $\mathbb{R}$, but T is not a linear transformation when $\mathbb{C}$ is viewed as a vector space over $\mathbb{C}$.