MATH 700
HOMEWORK 2

Due Friday, September 5, 2003 at the beginning of class

1. (Hoffman and Kunze, page 55, number 6.) Let V be the vector space over the complex numbers of all functions from $\mathbb{R}$ to $\mathbb{C}$. Let $f_1(x) = 1$, $f_2(x) = e^{\imath x}$, and $f_3(x) = e^{-\imath x}$.

   (a) Prove that f_1, f_2, f_3 are linearly independent.

- Suppose $f(x) = \sum\limits_{j=1}^{3} c_j f_j(x)$ is the zero function for some complex numbers c_j. Plug $x = 0$ into $f(x)$, $f'(x)$, and $f''(x)$ to see that

$$[\,c_1 \quad c_2 \quad c_3\,] \begin{bmatrix} 1 & 1 & 1 \\ 0 & \imath & -\imath \\ 0 & -1 & -1 \end{bmatrix} = [\,0 \quad 0 \quad 0\,].$$

   (b) Let $g_1(x) = 1$, $g_2(x) = \cos x$, and $g_3(x) = \sin x$. Find an invertible matrix P such that

$$[\,f_1 \quad f_2 \quad f_3\,]\,P = [\,g_1 \quad g_2 \quad g_3\,].$$

- Recall that $e^{\imath x} = \cos x + \imath \sin x$ for all real numbers theta. (If this identity is not familiar to you, you might want to think about the power series expansions of each side.) It follows that $e^{-\imath x} = \cos x - \imath \sin x$. Add the two expressions to learn that

$$\frac{e^{\imath x} + e^{-\imath x}}{2} = \cos x.$$

Subtract to learn that

$$\frac{e^{\imath x} - e^{-\imath x}}{2\imath} = \sin x.$$

So we take P to be

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2\imath} \\ 0 & \frac{1}{2} & -\frac{1}{2\imath} \end{bmatrix}.$$

2. Let V be a vector space of arbitrary dimension over the field F; let B be a basis for V; and let S be a linearly independent subset of V. Prove that there exists a subset S_1 of B such that $S \cup S_1$ is a basis for V.

- If S is already a basis for V, then there is nothing to prove. Otherwise, let P be the poset:

$$P = \{T \mid T \subseteq B \text{ and } S \cup T \text{ is linearly independent}\}.$$

The relation on P is $\subseteq$. The first sentence shows that P is non-empty. Let $\{T_i \mid i \in I\}$ be a chain in P. Observe that $T = \bigcup_{i \in I} T_i$ is an upper bound in P of the chain. (We know that T is in P because if $T \cup S$ were linearly dependent, then the offensive relation would involve elements of $T_i \cup S$ for some i.) We may now apply Zorn's Lemma to obtain a maximal element S_1 of P. I claim that $S \cup S_1$ is a basis for V. Otherwise, there is an element $b \in B$ with b not in the span of $S \cup S_1$. It follows that $S_1 \cup \{b\}$ is an element of P which properly contains the maximal element S_1. This is a contradiction.

3. Let V be the vector space of all polynomials in the variables $X_1, \dots, X_n$ over the field F.
   a. What is the dimension of the subspace W of V, which consists of all homogeneous polynomials of degree d?

- We must count the number of monomials of degree d in n variables. The answer is $\binom{n+d-1}{d}$. One can prove this by induction. A constructive proof is to count the number of work orders which consist of d picks and $n-1$ switches. Given such a work order the monomial builder goes to the workroom and steps up to the x_1 bin. The worker either picks up an x_1 or switches to the x_2 bin, etc., until the entire degree d monomial is made. (Combinatorists call this combinations with repetition allowed.)

   b. What is the dimension of the subspace W' of V, which consists of all polynomials of degree at most d?

- We must count the number of monomials of degree at most d in n variables. This is equivalent to counting the monomials of degree d in $n+1$ variables. (Use a fake variable to fill any missing degree.) So the answer is $\binom{n+d}{d}$.