

MATH 700
HOMEWORK 1 SOLUTIONS

Due Friday, August 29, 2003 at the beginning of class

1. (Hoffman and Kunze, page 49, number 7b.) Let V be the vector space of all 2×2 matrices over the field F . Let W_1 be the subspace of matrices of the form

$$\begin{bmatrix} x & -x \\ y & z \end{bmatrix}$$

and W_2 be the subspace of matrices of the form

$$\begin{bmatrix} a & b \\ -a & c \end{bmatrix}.$$

Find the dimensions of W_1 , W_2 , $W_1 + W_2$, and $W_1 \cap W_2$.

- The matrices

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

are a basis for W_1 ; so $\dim W_1 = 3$.

- The matrices

$$\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

are a basis for W_2 ; so $\dim W_2 = 3$.

- The matrices

$$\begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

are linearly independent and in $W_1 \cap W_2$. So

$$2 \leq \dim W_1 \cap W_2 \leq \dim W_2 = 3.$$

On the other hand, $W_1 \cap W_2 \neq W_2$, since $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in W_2 \setminus W_1$; hence $\dim W_1 \cap W_2 \neq 3$ and therefore $\dim W_1 \cap W_2 = 2$.

- It is clear that $W_1 + W_2 = V$ because

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

are in $W_1 + W_2$ and are a basis for V . Thus, $\dim W_1 + W_2 = 4$.

2. (Hoffman and Kunze, page 49, number 8.) Let V be the vector space of all 2×2 matrices over the field F . Find a basis A_1, A_2, A_3, A_4 for V such that $A_j^2 = A_j$ for each j .

- The matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

form one such basis.

3. (Hoffman and Kunze, page 49, number 14.) Let V be the set of real numbers. Regard V as a vector space over the field of **rational** numbers. Prove that this vector space is not finite-dimensional.

- Every finite dimensional vector space over \mathbb{Q} is countable; but V is not countable; hence, V is an infinite dimensional vector space over \mathbb{Q} .

4. Let V be the the vector space of rational functions in one variable over the field of complex numbers. (A typical element of V has the form $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomials with complex coefficients in one variable.) Prove that the dimension of V over \mathbb{C} is not countable.

- We will show that

$$\left\{ \frac{1}{x-a} \mid a \in \mathbb{C} \right\}$$

is linearly independent over \mathbb{C} . Suppose c_1, \dots, c_n and a_1, \dots, a_n are complex numbers, with the a_j distinct, and $\sum_{j=1}^n \frac{c_j}{x-a_j} = 0$. Multiply the most recent equation by the common denominator to see that

$$c_1(x-a_2) \cdots (x-a_n) + \text{a polynomial which vanishes at } a_1$$

is equal to the zero polynomial. Plug in a_1 for x to see that c_1 times a non-zero complex number is equal to zero. We conclude that $c_1 = 0$. Repeat the process to see that $c_1 = \cdots = c_n = 0$.