MATH 700
HOMEWORK 10 SOLUTIONS

Due Friday, December 5, 2003 at the beginning of class.

1. **Let $(M, +)$ be an abelian group generated by m_1, m_2, and m_3. Suppose that the maps**

$$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{g} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{f} M \to 0$$

form an exact sequence of abelian groups, where

$$f\left(\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}\right) = \sum_{i=1}^{3} n_i m_i \quad \textbf{and} \quad g\left(\begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{bmatrix}\right) = A \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{bmatrix}$$

for the matrix

$$A = \begin{bmatrix} 2 & 7 & 2 & 5 \\ 1 & 2 & 1 & 1 \\ 3 & 12 & 15 & 9 \end{bmatrix}.$$

Find a generating set m_1', m_2', m_3' for M so that M is equal to the internal direct sum of the cyclic groups $\mathbb{Z}m_1'$, $\mathbb{Z}m_2'$, and $\mathbb{Z}m_3'$. (Express the generators m_1', m_2', m_3' in terms of the original generators m_1, m_2, and m_3.) How many elements are in each subgroup $\mathbb{Z}m_i'$?

Consider the generating set $m_1' = m_2+2m_1+3m_3$, $m_2' = m_1+2m_3$, and $m_3' = m_3$ for M. Observe that $m_1' = 0$ (so $\mathbb{Z}m_1'$ has 0 elements), $\mathbb{Z}m_2'$ has 3 elements, and $\mathbb{Z}m_3'$ has 12 elements. Indeed,

$$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 7 & 2 & 5 \\ 3 & 12 & 15 & 9 \end{bmatrix}} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\begin{bmatrix} m_2 & m_1 & m_3 \end{bmatrix}} M \to 0,$$

$$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 3 \\ 3 & 6 & 12 & 6 \end{bmatrix}} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\begin{bmatrix} m_2 & m_1 & m_3 \end{bmatrix}} M \to 0,$$

$$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 3 \\ 0 & 6 & 12 & 6 \end{bmatrix}} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\begin{bmatrix} m_2+2m_1+3m_3 & m_1 & m_3 \end{bmatrix}} M \to 0,$$

$$\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\begin{bmatrix}1&0&0&0\\0&3&0&0\\0&6&12&0\end{bmatrix}} \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\begin{bmatrix}m_2+2m_1+3m_3 & m_1 & m_3\end{bmatrix}} M \to 0,$$

and

$$\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\begin{bmatrix}1&0&0&0\\0&3&0&0\\0&0&12&0\end{bmatrix}} \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\begin{bmatrix}m_2+2m_1+3m_3 & m_1+2m_3 & m_3\end{bmatrix}} M \to 0$$

all are exact.

2. **Let B be an $n\times n$ matrix over the field F, M be the vector space F^n with standard basis $e_1,\dots,e_n$, and R be the polynomial ring $F[x]$. View M as an R-module by way of the action $xv = Bv$ for all $v\in M$. Find an $n\times p$ matrix A, with entries in R, for some p, such that**

$$R^p \xrightarrow{g} R^n \xrightarrow{f} M \longrightarrow 0$$

is an exact sequence of R-modules where

$$f\left(\begin{bmatrix}r_1\\ \vdots\\ r_n\end{bmatrix}\right) = \sum_{i=1}^n r_ie_i \quad \textbf{and} \quad g\left(\begin{bmatrix}r_1\\ \vdots\\ r_p\end{bmatrix}\right) = A\begin{bmatrix}r_1\\ \vdots\\ r_p\end{bmatrix}$$

for all

$$\begin{bmatrix}r_1\\ \vdots\\ r_n\end{bmatrix}\in R^n \quad \textbf{and} \quad \begin{bmatrix}r_1\\ \vdots\\ r_p\end{bmatrix}\in R^p.$$

Give the matrix A explicitly. Prove your answer.

Let $p=n$ and let $\boxed{A = xI - B}$. We first show that $\operatorname{im} g \subseteq \ker f$. Let $w = \begin{bmatrix}0\\ \vdots\\ 0\\ 1\\ 0\\ \vdots\\ 0\end{bmatrix}$ in R^p be the vector with 1 in position j and zeros everywhere else. Then

$$f(g(w)) = f\left(\begin{bmatrix}0\\ \vdots\\ 0\\ x\\ 0\\ \vdots\\ 0\end{bmatrix} - \begin{bmatrix}b_{1j}\\ \vdots\\ b_{nj}\end{bmatrix}\right) = xe_j - \sum_i b_{ij}e_i = Be_j - \sum_i b_{ij}e_i = 0.$$

Now we show that $\ker f \subseteq \operatorname{im} g$. Let $u \in R^n$ be an element of $\ker f$. There exists d such that

$$u = x^d u_d + x^{d-1} u_{d-1} + \dots x^1 u_1 + u_0,$$

where each u_k is an element of F^p. Observe that every entry of $u - g(x^{d-1} u_d)$ has degree at most $d-1$. Iterate this procedure to obtain $w \in R^p$ with the property that every entry of $u - g(w)$ is a constant. Observe that, $u - g(w)$ is still in the kernel of f. Observe also that the only vector of constants which is in the kernel of f is the zero vector, since $e_1, \dots, e_n$ are linearly independent elements of M. It follows that $u = g(w)$.