# MATH 700
# HOMEWORK 10

Due Friday, December 5, 2003 at the beginning of class.

1. Let $(M,+)$ be an abelian group generated by $m_1$, $m_2$, and $m_3$. Suppose that the maps

$$\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \xrightarrow{g} \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \xrightarrow{f} M \to 0$$

form an exact sequence of abelian groups, where

$$f\left(\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}\right) = \sum_{i=1}^{3} n_i m_i \quad \text{and} \quad g\left(\begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{bmatrix}\right) = A \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{bmatrix}$$

for the matrix

$$A = \begin{bmatrix} 2 & 7 & 2 & 5 \\ 1 & 2 & 1 & 1 \\ 3 & 12 & 15 & 9 \end{bmatrix}.$$

Find a generating set $m_1', m_2', m_3'$ for $M$ so that $M$ is equal to the internal direct sum of the cyclic groups $\mathbb{Z}m_1'$, $\mathbb{Z}m_2'$, and $\mathbb{Z}m_3'$. (Express the generators $m_1', m_2', m_3'$ in terms of the original generators $m_1$, $m_2$, and $m_3$.) How many elements are in each subgroup $\mathbb{Z}m_i'$?

2. Let $B$ be an $n \times n$ matrix over the field $F$, $M$ be the vector space $F^n$ with standard basis $e_1, \ldots, e_n$, and $R$ be the polynomial ring $F[x]$. View $M$ as an $R$-module by way of the action $xv = Bv$ for all $v \in M$. Find an $n \times p$ matrix $A$, with entries in $R$, for some $p$, such that

$$R^p \xrightarrow{g} R^n \xrightarrow{f} M \to 0$$

is an exact sequence of $R$-modules where

$$f\left(\begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}\right) = \sum_{i=1}^{n} r_i e_i \quad \text{and} \quad g\left(\begin{bmatrix} r_1 \\ \vdots \\ r_p \end{bmatrix}\right) = A \begin{bmatrix} r_1 \\ \vdots \\ r_p \end{bmatrix}$$

for all

$$\begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} \in R^n \quad \text{and} \quad \begin{bmatrix} r_1 \\ \vdots \\ r_p \end{bmatrix} \in R^p.$$

Give the matrix $A$ explicitly. Prove your answer.