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Quiz for February 2, 2006

Let S be a set of $n+1$ integers between 1 and $2n$. Prove that at least one integer from S divides another integer from S.

ANSWER: We will prove the statement by induction on n.

Base case: If $n=1$, then S consists of two numbers from $\{1,2\}$; so, $S=\{1,2\}$ and one of the integers from S (namely 1) does indeed the other integer from S (namely 2).

Inductive step: Let n be some fixed integer with $2 \le n$. We suppose that the statement holds for $n-1$. We prove that the statement holds at n.

We finish the argument by contradiction. Suppose that there exists a counter example to the statement at n. That is, suppose that S consists of $s_1 < \cdots < s_{n+1}$, with $1 \le s_1$ and $s_{n+1} \le 2n$; but s_i does not divide s_j for any $i<j$. We will produce a counter example to the statement at $n-1$.

If $s_n \le 2(n-1)$, then $\{s_1, \dots, s_n\}$ is a counter example to the statement at $n-1$. The induction hypothesis tells us that the statement holds at $n-1$; so we know that

$$2n-1 \le s_n < s_{n+1} \le 2n.$$

Thus, we know that

$$2n-1 = s_n \quad \text{and} \quad 2n = s_{n+1}.$$

If $i \le n-1$, then s_i does not divide $s_{n+1} = 2n$. Thus, none of the numbers $s_1, \dots, s_{n-1}$ is equal to n and none of these numbers divide n. Furthermore, all of the numbers $s_1, \dots, s_{n-1}$ are less than $2n$ so n does not divide any of these numbers. We see that the set of numbers

$$T = \{s_1, \dots, s_{n-1}\} \cup \{n\}$$

is a counter example to the statement at $n-1$. (In other words, T is a set of $(n-1)+1$ numbers between 1 and $2(n-1)$ and none of the numbers in T divide any of the other numbers in T.) The existence of T contradicts the Inductive hypothesis. This is a contradiction. Our supposition (that there exists a counter example to the statement at n) must be false. In other words, if the original statement holds at $n-1$, then the original statement also holds at n. The proof of the inductive step is complete; and therefore, the proof is complete.