Math 574, Exam 2, Summer 2007 Solutions

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

Please leave room in the upper left corner for the staple.

There are 9 problems **ON TWO SIDES!**. The exam is worth a total of 50 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

You should **KEEP** this copy of your exam.

I will post the solutions on my website sometime after 3:15 today.

1. (6 points) **Simplify** $\sum\limits_{k=0}^{n} \binom{n}{k}^2$ **. Explain thoroughly.**

This sum is equal to $\boxed{\binom{2n}{n}}$. The expression is equal to

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}.$$

Suppose you have n men and n women. The above sum counts ALL n-element subsets of your people. A different way to make the same count is $\binom{2n}{n}$.

2. (6 points) **Simplify** $\binom{50}{0} + \binom{51}{1} + \binom{52}{2} + \cdots + \binom{200}{150}$ **. Explain thoroughly.**

We proved in class –twice – that

$$\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r}.$$

In this problem $n = 50$ and $r = 150$; so the sum is $\boxed{\binom{201}{150}}$.

3. (6 points) **Simplify** $\sum_{k=0}^{n} k\binom{n}{k}$ **. Explain thoroughly.**

How many ways can one pick a subcommittee from an n-element committee, and then pick a chair of this subcommittee? The given formula picks the subcommittee first. One can also do the problem by picking the chair (there are n ways to do this) and then the rest of the subcommittee. (There are 2^{n-1} ways to pick a subset out of an $n-1$ element set.) The given expression is equal to $\boxed{n2^{n-1}}$.

4. (6 points) **Seven women and nine men are on the faculty in the Mathematics department at a school. How many ways are there to select a committee of five members if at least one woman must be on the committee?**

There are $\binom{16}{5}$ ways to pick a five member committee out of a 16 member set; $\binom{9}{5}$ of these committes consist of all men. So there are

$$\boxed{\binom{16}{5} - \binom{9}{5}}$$

ways to select a committee of five members if at least one woman must be on the committee?

5. (6 points) **List the numbers from 1 to 16 in such a way that no sublist of size 5 is strictly increasing or strictly decreasing.**

$$\boxed{13, 9, 5, 1, 14, 10, 6, 2, 15, 11, 7, 3, 16, 12, 8, 4}$$

6. (5 points) **Consider a group of 20 people where any two people are either friends or enemies. Prove that there are either four mutual friends or four mutual enemies.**

NOTE: On quiz two you already proved that in any group of 10 people (where any two people are either friends or enemeies), there are either three mutual friends or four mutual enemies. The same work as on quiz two would prove that in any group of 10 people (where any two people are either friends or enemeies), there are either three mutual enemies or four mutual friends. You need not prove either of these results any further and you may use these results in today's question.

Let A be one of the 20 people. Partition the remaining 19 people into A's friends and A's enemies. The pigeon hole principal ensures that A has at least 10 friends or A has at least 10 enemeies.

Case 1 A has at least 10 friends. Quiz 2 ensures that among A's friends there are either three mutual friends or four mutual enemeies.

Case 1a A has at least 10 friends and among A's friends there are either three mutual friends. In this case, A together with these 3 friends forms a clique of 4 mutual friends.

Case 1b A has at least 10 friends and among A's friends there are four mutual enemeies. We have a clique of four mutual enemies.

Case 2 A has at least 10 enemies. The problem which is symmetric to Quiz 2 ensures that among A's enemies there are either three mutual enemies or four mutual friends.

Case 2a A has at least 10 enemies and among A's enemies there are three mutual enemies. In this case, A together with these 3 enemies forms a clique of 4 mutual enemies.

Case 2b A has at least 10 enemies and among A's enemies there are four mutual friends. We have a clique of four mutual friends.

7. (5 points) **In how many ways can a photographer at a wedding arrange six people in a row, including the bride and groom, if the bride must be next to the groom.**

Think of the bride-groom as one unit. There are $5!$ ways to arrange b-g and the other four people. There are another $5!$ ways to arrange g-b and the other four people. In total there are $\boxed{2(5!)}$ possible arrangements.

8. (5 points) **Recall that the Fibonacci numbers are:** $f_1 = 1$ **,** $f_2 = 1$ **, and for** $n \geq 3$ $f_n = f_{n-1} + f_{n-2}$ **. Prove that** $f_1^2 + f_2^2 + f_3^2 + \cdots + f_n^2 = f_n f_{n+1}$ **whenever** n **is a positive integer.**

The proof is by induction on n.

Base Case If $n = 1$, then the left side is $f_1^2 = 1$ and the right side is $f_1 f_2 = 1$.

Induction Hypothesis. Assume that $f_1^2 + f_2^2 + f_3^2 + \cdots + f_n^2 = f_n f_{n+1}$ **for some fixed** n **.**

We must show: $f_1^2 + f_2^2 + f_3^2 + \cdots + f_{n+1}^2 = f_{n+1} f_{n+2}$.

The left side of the proposed identity is $\left(f_1^2 + f_2^2 + f_3^2 + \cdots + f_n^2\right) + f_{n+1}^2$ and by induction this is

$$f_n f_{n+1} + f_{n+1}^2 = f_{n+1}(f_n + f_{n+1}) == f_{n+1} f_{n+2},$$

as desired.

9. (5 points) **Show that**

$$\sum_{\{a_1,\ldots,a_k\}\subseteq\{1,\ldots,n\}} \frac{1}{a_1 a_2 \cdots a_k} = n.$$

(Here the sum is taken over all nonempty subsets of the n smallest positive integers.)

The proof is by induction on n.

Base Case If $n = 1$, then both sides are 1.

Induction Hypothesis. Assume that

$$\sum_{\{a_1,\ldots,a_k\}\subseteq\{1,\ldots,n\}} \frac{1}{a_1 a_2 \cdots a_k} = n,$$

for some fixed positive integer n.

We must show:

$$\sum_{\{a_1,\ldots,a_k\}\subseteq\{1,\ldots,n+1\}} \frac{1}{a_1 a_2 \cdots a_k} = n+1.$$

We attack the left side of the proposed equation. We notice that there are three types of nonempty subsets of $\{1,\ldots,n+1\}$. Some such subsets do not contain $n+1$; some such subsets include $n+1$ and nothing else; some such subsets are the union $\{n+1\}\cup S$ where S is a nonempty subset of $\{1,\ldots,n\}$. It follows that the left side is

$$\sum_{\{a_1,\ldots,a_k\}\subseteq\{1,\ldots,n\}} \frac{1}{a_1 a_2 \cdots a_k} + \frac{1}{n+1} + \frac{1}{n+1} \sum_{\{a_1,\ldots,a_k\}\subseteq\{1,\ldots,n\}} \frac{1}{a_1 a_2 \cdots a_k}.$$

Use the induction hypothesis, twice, to see that the left side is

$$n + \tfrac{1}{n+1} + \tfrac{n}{n+1} = n + \tfrac{n+1}{n+1} = n+1,$$

as desired.