Math 574, Exam 1, Summer 2007 Solutions

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

Please leave room in the upper left corner for the staple.

There are 8 problems **ON TWO SIDES!**. The exam is worth a total of 50 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will post the solutions on my website sometime after 3:15 today.

1. (7 points) **Suppose that $A_1, \dots, A_n$ are sets where $n \geq 2$. Suppose also that for all pairs of integers i and j with $1 \leq i < j \leq n$, either $A_i \subseteq A_j$ or $A_j \subseteq A_i$. Prove that there exists an integer i, with $1 \leq i \leq n$, such that $A_i \subseteq A_j$ for all j with $1 \leq j \leq n$.**

Proof by induction.

Base Case: The listed statement is clear when $n = 2$.

Inductive Hypothesis: We assume that the listed statement holds for some FIXED n.

We must show: the listed statement holds when all of the n's are replaced by $n+1$.

We are given $n+1$ sets $A_1, \dots, A_{n+1}$. We are told that for all pairs of integers i and j, with $1 \leq i < j \leq n+1$, either $A_i \subseteq A_j$ or $A_j \subseteq A_i$.

For the time being we ignore set A_{n+1}. We may apply the inductive hypothesis to the sets $A_1, \dots, A_n$. Thus, there exists a subscript i_0 with $1 \leq i_0 \leq n$ such that $A_{i_0} \subseteq A_j$ for all j with $1 \leq j \leq n$. Now we compare A_{i_0} and A_{n+1}. The hypothesis tells us that either $A_{i_0} \subseteq A_{n+1}$ or $A_{n+1} \subseteq A_{i_0}$. In the first case , we let $i = i_0$. In the second case, we let $i = n+1$. In each case, we have $A_i \subseteq A_j$ for all j with $1 \leq j \leq n+1$. The proof is complete.

2. (7 points) **Prove that if n is a positive integer, then 133 divides $11^{n+1} + 12^{2n-1}$.**

Proof by induction.

Base Case: We prove that 133 divides $11^{n+1} + 12^{2n-1}$, when $n = 1$. In this case, $11^{n+1} + 12^{2n-1} = 11^2 + 12^1 = 121 + 12 = 133$. It is certainly true that 133 divides 133.

Inductive Hypothesis: Assume that 133 divides $11^{n+1} + 12^{2n-1}$ for some FIXED n.

We must show: that 133 divides $11^{n+2} + 12^{2n+1}$.

We see that

$$11^{n+2} + 12^{2n+1} = (11)11^{n+1} + (144)12^{2n-1} = 11(11^{n+1} + 12^{2n-1}) + (133)12^{2n-1}.$$

The inductive hypothesis tells us that 133 divides the first term. It is obvious that 133 divides the second term. We conclude that 133 divides $11^{n+2} + 12^{2n+1}$, and this is what we were supposed to show. The proof is complete.

3. (6 points) **Prove that** $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$**, whenever** n **is a positive integer.**

Proof by induction.

Base Case: We prove the listed formula at $n = 1$. When $n = 1$, the left side is $1 \cdot 1! = 1$ and the right side is $2! - 1 = 1$. The formula holds at $n = 1$.

Inductive Hypothesis: Assume that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$ for some FIXED positive integer n.

We must show:

$$(***) \qquad 1 \cdot 1! + 2 \cdot 2! + \cdots + (n+1) \cdot (n+1)! = (n+2)! - 1.$$

We see that the left side of (***) is

$$(1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n!) + (n+1) \cdot (n+1)!.$$

The inductive hypothesis shows that this is equal to

$$(n+1)! - 1 + (n+1) \cdot (n+1)! = (n+1)![1 + (n+1)] - 1 = (n+2)! - 1,$$

which is the right side of (***). We have established (***). The proof is complete.

4. (6 points) **Let f be a function from the real numbers to the real numbers, and let a be a real number. What is the negation of the statement: "For all real numbers $\varepsilon > 0$, there exists a real number $\delta > 0$, such that if x is a real number, with $0 < |x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$"?**

There exists a real number $\varepsilon > 0$, such that for all real numbers $\delta > 0$, there exists a real number x with $0 < |x - a| < \delta$ but $|f(x) - f(a)| \geq \varepsilon$.

5. (6 points) **Let A, B, and C be sets, and $g\colon A \to B$ and $f\colon B \to C$ be functions. Suppose that f is onto and $f \circ g$ is onto. Does g have to be onto? If yes, prove your answer. If no, give a counterexample.**

NO. Suppose $A = C = \{1\}$, $B = \{1, 2\}$, $g(1) = 1$, $f(1) = f(2) = 1$. We see that A, B, and C are sets, f and g are functions, f is onto and $f \circ g$ is onto; however, g is not onto because $g(a) \neq 2$ for any $a \in A$.

6. (6 points) **List the elements of $\mathfrak{P}(\mathfrak{P}(\emptyset))$. (In this problem, if S is a set, then $\mathfrak{P}(S)$ is the power set of S.)**

We see that $\mathfrak{P}(\emptyset) = \{\emptyset\}$; and therefore, the elements of $\mathfrak{P}(\mathfrak{P}(\emptyset))$ are:

$$\emptyset, \{\emptyset\}.$$

7. (6 points) **Let $A_i = \{\ldots, -2, -1, 0, 1, \ldots, i\}$. Find**
   (a) $\bigcup\limits_{i=1}^{n} A_i$**, and**
   (b) $\bigcap\limits_{i=1}^{n} A_i$**.**

We see that
   (a) $\bigcup\limits_{i=1}^{n} A_i = A_n$, and
   (b) $\bigcap\limits_{i=1}^{n} A_i = A_1$.

8. (6 points) **Consider the statement "if $3 < x$, then $9 < x^2$".**
   (a) What is the converse of the original statement?
   (b) Is (a) logically equivalent to the original statement?
   (c) What is the contrapositive of the original statement?
   (d) Is (c) logically equivalent to the original statement?

   (a) The converse of the original statement is "if $9 < x^2$, then $3 < x$".

(b) NO.
(c) The contrapositive of the original statement is "if $9 \geq x^2$, then $3 \geq x$.
(d) YES.