Math 574, Final Exam, Spring 2006

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

There are 11 problems. Problems 1 through 10 are worth 9 points each. Problem 11 is worth 10 points. The exam is worth 100 points.

YOU MUST JUSTIFY YOUR ANSWERS. Write in complete sentences. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will post the solutions on my website a few hours after the exam is finished.

1. Express the sum $\sum_{k=0}^{n} \binom{n}{k}$ in a closed form.

2.
   (a) Consider the list of numbers

$$a_1 = 4, \quad a_2 = 6, \quad a_3 = 2, \quad a_4 = 8, \quad a_5 = 10, \quad a_6 = 1, \quad a_7 = 5,$$
$$a_8 = 9, \quad a_9 = 7, \quad a_{10} = 3.$$

   For each integer i with $1 \le i \le 10$, let u_i be the length of the longest increasing sequence from the above list which starts at a_i, and let d_i be the length of the longest decreasing sequence from the above list which starts at a_i. **Write down** the value of (u_i, d_i) for each i.

   (b) Let $a_1, \dots, a_{10}$ be any list of 10 distinct numbers. Define (u_i, d_i) as in part (a). **Prove** that if $i < j$, then $(u_i, d_i) \ne (u_j, d_j)$.

   (c) Prove that every list $a_1, \dots, a_{10}$ of 10 distinct numbers must contain an increasing sublist of length 4 or a decreasing sublist of length 4.

   (d) Give an example of a list $a_1, \dots, a_9$ of 9 distinct numbers which does not contain an increasing sublist of length 4 or a decreasing sublist of length 4.

3.
   (a) What is the truth table for $p \to q$?

   (b) What is the converse of $p \to q$?

   (c) What is the contrapositive of $p \to q$?

   (d) Is the converse of $p \to q$ logically equivalent to $p \to q$?

   (e) Is the contrapositive of $p \to q$ logically equivalent to $p \to q$?

   (f) Express $p \to q$ in a logically equivalent manner using only $\wedge$, $\vee$, and "not".

4. Let I be the following interval of real numbers: $I = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$. For each real number x in I, let S_x be the following set of real numbers:

$$S_x = \{y \in \mathbb{R} \mid x - \tfrac{3}{4} < y < x + \tfrac{3}{4}\}.$$

   (a) Find $\bigcup_{x \in I} S_x$.

   (b) Find $\bigcap_{x \in I} S_x$.

5. How many words of length 20 can be made from the alphabet $\{0, 1, 2, 3\}$ if exactly 10 zeros are used?

6. Prove that every integer greater than 11 is the sum of 2 composite numbers.

7. Let S, T, and U be sets, and let $f\colon S \to T$ and $g\colon T \to U$ be functions. Suppose that $g \circ f$ is onto. For each question, prove or give a counterexample.
   (a) Does f have to be onto?
   (b) Does g have to be onto?

8. Recall that the Fibonacci numbers are: $f_1 = 1$, $f_2 = 1$, and for each integer n with $n \geq 3$, $f_n = f_{n-1} + f_{n-2}$. Prove that f_{4n} is a multiple of 3, whenever n is a positive integer.

9. How many monomials of degree less than or equal to d can be made using the n variables $x_1, \ldots, x_n$? (For example, $x_1^2 x_2^3$ is a monomial of degree 5.)

10. Find a recurrence relation for the number of strings made from 0's, 1's, and 2's that do not contain two consecutive zeros or two consecutive ones.

11. Solve the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2} + 2^n$ with $a_0 = 1$ and $a_1 = 7$. **CHECK your answer.**