

11. (14 points) Find a CLOSED formula for $\sum_{k=0}^n k^4$.

Let $a_n = \sum_{k=0}^n k^4$. We first solve $a_n = a_{n-1} + n^4$, $a_0 = 0$

Solve $a_n = a_{n-1}$ is $a_n = C_1$

A particular solution of $a_n = a_{n-1} + n^4$ has the form $a_n = An^5 + Bn^4 + Cn^3 + Dn^2 + En$.

$$An^5 + Bn^4 + Cn^3 + Dn^2 + En = n^4 + A(n-1)^5 + B(n-1)^4 + C(n-1)^3 + D(n-1)^2 + E(n-1)$$

$$An^5 + Bn^4 + Cn^3 + Dn^2 + En = n^4 + A[n^5 - 5n^4 + 10n^3 - 10n^2 + 5n - 1] \\ + B[n^4 - 4n^3 + 6n^2 - 4n + 1] \\ + C[n^3 - 3n^2 + 3n - 1] \\ + D[n^2 - 2n + 1] \\ + E[n - 1]$$

$$0 = 1 - 5A$$

$$0 = 10A - 4B$$

$$0 = -10A + 6B - 3C$$

$$0 = 5A - 4B + 3C - 2D$$

$$0 = -A + B - C + D - E$$

$$A = \frac{1}{5}$$

$$B = \frac{10}{4} \left(\frac{1}{5}\right) = \frac{1}{2}$$

$$C = \frac{1}{3} \left[-10\left(\frac{1}{5}\right) + 6\left(\frac{1}{2}\right) \right] = \frac{1}{3} [-2 + 3] = \frac{1}{3}$$

$$D = \frac{1}{2} \left[5\left(\frac{1}{5}\right) - 4\left(\frac{1}{2}\right) + 3\left(\frac{1}{3}\right) \right] = \frac{1}{2} [1 - 2 + 1] = 0$$

$$E = -\frac{1}{5} + \frac{1}{2} - \frac{1}{3} + 0 = \frac{1}{30} [-6 + 15 - 10] = -\frac{1}{30}$$

So gen sol of $a_n = a_{n-1} + n^4$ is

$$a_n = C_1 + \frac{1}{30} [6n^5 + 15n^4 + 10n^3 - n]$$

$$0 = C_1 + 0 \quad \therefore C_1 = 0$$

$$\therefore \sum_{k=0}^n k^4 = \frac{n(n+1)(6n^3 + 9n^2 + n - 1)}{30}$$

6 is good idea, but
it doesn't work.