11. (14 points) Find a CLOSED formula for $\sum_{k=0}^{n} k^4$.

Let $a_n = \sum_{k=0}^{n} k^4$. We must solve $a_n = a_{n-1} + n^4$, $a_0 = 0$

Sol of $a_n = a_{n-1}$ is $a_n = C_1$

A particular solution of $a_n = a_{n-1} + n^4$ has the form $a_n = An^5 + Bn^4 + Cn^3 + Dn^2 + En$.

$An^5 + Bn^4 + Cn^3 + Dn^2 + En = n^4 + A(n-1)^5 + B(n-1)^4 + C(n-1)^3 + D(n-1)^2 + E(n-1)$

$\cancel{An^5} + \cancel{Bn^4} + \cancel{Cn^3} + \cancel{Dn^2} + \cancel{En} = n^4 + A[\cancel{n^5} - 5n^4 + 10n^3 - 10n^2 + 5n - 1]$

$+ B[\cancel{n^4} - 4n^3 + 6n^2 - 4n + 1]$

$+ C[\cancel{n^3} - 3n^2 + 3n - 1]$

$+ D[\cancel{n^2} - 2n + 1]$

$+ E[\cancel{n} - 1]$

$0 = 1 - 5A$

$0 = 10A - 4B$

$0 = -10A + 6B - 3C$

$0 = 5A - 4B + 3C - 2D$

$0 = -A + B - C + D - E$

$A = \frac{1}{5}$

$B = \frac{10}{4}\left(\frac{1}{5}\right) = \frac{1}{2}$

$C = \frac{1}{3}\left[-10\left(\frac{1}{5}\right) + 6\left(\frac{1}{2}\right)\right] = \frac{1}{3}\left[-2+3\right] = \frac{1}{3}$

$D = \frac{1}{2}\left[5\left(\frac{1}{5}\right) - 4\left(\frac{1}{2}\right) + 3\left(\frac{1}{3}\right)\right] = \frac{1}{2}\left[1 - 2 + 1\right] = 0$

$E = -\frac{1}{5} + \frac{1}{2} - \frac{1}{3} + 0 = \frac{1}{30}\left[-6 + 15 - 10\right] = -\frac{1}{30}$

so gen sol of $a_n = a_{n-1} + n^4$ is

$a_n = C_1 + \frac{1}{30}\left[6n^5 + 15n^4 + 10n^3 - n\right]$

$0 = C_1 + 0 \quad \therefore C_1 = 0$

$\therefore \sum_{k=0}^{n} k^4 = \frac{n(n+1)(6n^3 + 9n^2 + n - 1)}{30}$

6 is good idea, but it doesn't corr.