Math 574, Exam 3, Solutions, Spring 2006

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

There are 8 problems. Problems 1 and 2 are worth 7 points each. Each of the other problems is worth 6 points. SHOW your work. Make your work be coherent and clear. Write in complete sentences whenever this is possible. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

YOU MUST JUSTIFY YOUR ANSWERS.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will post the solutions on my website a few hours after the exam is finished.

1. **Prove that for each positive integer** n **,** $\sum\limits_{k=1}^{n}(2k-1)$ **is a perfect square.**

We prove that $\sum\limits_{k=1}^{n}(2k-1)=n^2$ by induction on n.

When $n=1$, both sides are 1.

INDUCTIVE HYPOTHESIS: Assume that $\sum\limits_{k=1}^{n}(2k-1)=n^2$ for some fixed positive integer n.

WE WILL PROVE: $\sum\limits_{k=1}^{n+1}(2k-1)=(n+1)^2$.

We see that $\sum\limits_{k=1}^{n+1}(2k-1)=\sum\limits_{k=1}^{n}(2k-1)+(2(n+1)-1)$ and the Inductive Hypothesis tells us that this is equal to:

$$n^2+2n+1=(n+1)^2.$$

We have shown that the inductive hypothesis ensures that $\sum\limits_{k=1}^{n+1}(2k-1)=(n+1)^2$, and our proof is complete.

2. **Recall that the Fibonacci numbers are:** $f_1=1$ **,** $f_2=1$ **, and for each integer** n **with** $n\geq 3$ **,** $f_n=f_{n-1}+f_{n-2}$ **. Prove that** f_{4n} **is a multiple of** 3 **, whenever** n **is a positive integer.**

The Fibonacci numbers are $f_1 = 1$, $f_2 = 1$, $f_3 = 2$, and $f_4 = 3$. We see that $f_{4 \cdot 1}$ is a multiple of 3 and this takes care of the base case. We continue by induction.

INDUCTIVE HYPOTHESIS: Assume that $f_{4n} = 3\ell$ for some fixed positive integers n and ℓ.

WE WILL PROVE: f_{4n+4} is also a multiple of 3.

We see that

$$f_{4n+4} = f_{4n+3} + f_{4n+2} = 2f_{4n+2} + f_{4n+1} = 3f_{4n+1} + 2f_{4n} = 3(f_{4n+1} + 2\ell).$$

We have shown that the inductive hypothesis ensures that f_{4n+4} is a multiple of 3, and our proof is complete.

In class we proved the following result. You may use this result in your solution to problem 3.

Result from class. *If each pair in a group of six people consists of two friends or two enemies, then there are three mutual friends or three mutual enemies in the group.*

3. **Consider a group of ten people in which each pair of individuals consists of two friends or two enemies. Prove that there are either three mutual friends or four mutual enemies in the group of ten people.**

Let A be one of the people. The Pigeon Hole Principle ensures that A has at least 4 friends or at least 6 enemies. (Indeed, if A has less than 4 friends, all of the non-friends, and there at least six of them, are enemies of A.) We will examine each case carefully.

Case 1. A has at least 4 friends $\{B, C, D, E\}$ AND at least one pair from $\{B, C, D, E\}$ are friends. We are finished in this case because A together with the pair of friends from $\{B, C, D, E\}$ form a triangle of mutual friends.

Case 2. A has at least 4 friends $\{B, C, D, E\}$ AND every pair from $\{B, C, D, E\}$ is a pair of enemies. We are finished in this case because $\{B, C, D, E\}$ is a four-some of mutual enemies.

Case 3. A has at least 6 enemies $\{F, G, H, I, J, K\}$. The result from class tells us that $\{F, G, H, I, J, K\}$ contains a triangle of mutual friends or a triangle of mutual enemies. Case 3 is the case where $\{F, G, H, I, J, K\}$ contains a triangle of mutual friends. Once again we are finished because we have exhibited a triangle of mutual friends.

Case 4. A has at least 6 enemies $\{F, G, H, I, J, K\}$, and the enemy set $\{F, G, H, I, J, K\}$ contains a triangle of mutual enemies. We are also finished in this case because A together with the three mutual enemies from $\{F, G, H, I, J, K\}$ forms a four-some of of mutual enemies.

4. **Express the sum**

$$\binom{500}{0}+\binom{501}{1}+\binom{502}{2}+\cdots+\binom{600}{100}$$

as a single binomial coefficient. PROVE your answer. (If you quote a formula we did in class, then the prove the formula.)

For each integer k between 0 and 100, the binomial coefficient

$$\binom{500+k}{k}$$

counts the number of strings which consists of 500 ones and k zeros. This binomial coefficient also counts the number of strings of 501 ones and 100 zeros where the right most 1 sits in position $500+k+1$. (If you have a string of 500 ones and k zeros, you convert this into a string of 501 ones and 100 zeros by putting 1 in the first unused spot and right justifying using zeros. Now we see that the given sum of binomial coefficients counts the number of strings which consists of 501 ones and 100 zeros. (Be sure to notice that each such string has been counted EXACTLY once.) The quickest way to make this count is

$$\boxed{\binom{601}{100}.}$$

5. **Express the sum**

$$\binom{20}{0}\binom{30}{10}+\binom{20}{1}\binom{30}{9}+\cdots+\binom{20}{9}\binom{30}{1}+\binom{20}{10}\binom{30}{0}$$

as a single binomial coefficient. PROVE your answer. (If you quote a formula we did in class, then the prove the formula.)

Consider a committee which consists of 50 people, 30 of the people are women and 20 are men. There are at least two ways to count the number of subcommittees which consist of 10 people. The easiest way to make this count is $\boxed{\binom{50}{10}}$.

Another way to make this count is given in the statement of the problem. The product of binomial coefficients $\binom{20}{k}\binom{30}{10-k}$ counts the number of subcommittees which contains k men and $10-k$ women. It follows that

$$\sum_{k=0}^{10}\binom{20}{k}\binom{30}{10-k}=\binom{50}{10}.$$

6. **How many paths are there from $(0,0)$ to $(6,4)$ on the xy-plane if each path consists of either of a series of steps, where each step is a move one unit to the right or a move one unit up. (No moves to the left or downward are allowed.)**

We need to count the number of words made with 6 r's and 4 u's. This is the same as the number of 6-member subcommittees of a 10-member committee: $\boxed{\binom{10}{6}}$.

7. **How many words of length 20 can be made from the alphabet $\{0,1,2,3\}$ if exactly 10 zeros are used?**

Select the 10 places to put zero. There are $\binom{20}{10}$ ways to do this. Now fill in the rest of the spots. There are 3^{10} ways to do this. The answer is

$$\boxed{3^{10}\binom{20}{10}}.$$

8. **Let S be the set of integers i with $1 \le i \le 33$. Find the smallest integer k with the property that every subset T of S of size k contains one integer which divides another integer from T. In other words, you must:**
   (a) tell me what k is,
   (b) prove that every subset T of S of size k contains one integer which divides another integer from T,
   (c) give an example of a $k-1$ element subset U of S in which no integer from U divides any other integer from U.

   (a) k is $\boxed{18}$,
   (c) We see that $U = \{17, 18, \ldots, 33\}$ is a set of 17 integers, all chosen between 1 and 33 and no number from U divides any other number in U.
   (b) Let T be a subset of S with 18 elements. We will prove that some element of T divides some other element of T. There are 17 odd integers o between 1 and 33. For each of these odd integers o, let S_o be

$$S \cap \{o2^k \mid \ k \text{ is a positive integer}\}.$$

   We see that S is the disjoint union of the sets S_o. The set T has 18 elements. The set T is contained in the union of the 17 sets S_o. So, the Pigeon Hole Principle tells us that T contains at least two elements from some S_o. We set the sets S_o up in such a way that if one takes two different elements $o2^k$ and $o2^K$ from the same S_o then the smaller numbers divides into the larger number. We conclude that T contains a number which divides some other number in T.