(a) For each graph G, let $\Delta(G) = \max\{\deg u \mid u \text{ is a vertex of } G\}$.

(a) Give an inequality (of the form ___ $\leq$ ___) which relates $\Delta(G)$ and $\chi(G)$.

$$\chi(G) \leq \Delta(G) + 1$$

(b) Give an example of a graph G such that your formula in (a) holds with $=$ in place of $\leq$.

$\chi = 1$

$\Delta = 0$

(c) Give an example of a graph G such that your formula in (b) holds with $<$ in place of $\leq$.

$\chi = 2$ $\quad$ $\chi = 2$ $\quad$ $\chi = 2$

$\Delta + 1 = 3$ $\quad$ $\Delta + 1 = 4$ $\quad$ $\Delta + 1 = 5$

(d) Prove that your formula in (a) holds for every Graph G.

Induct on the # of vertices of G.
If G has one vertex, then G is • and it works.

IH (suppose $\chi(G) \leq \Delta(G) + 1$ whenever # of vertices of $G < N$.)

Let G have N vertices. Let G' be G with one vertex u removed.

IH says $\chi(G') \leq \Delta(G') + 1$

But $\Delta(G') \leq \Delta(G)$

so $\chi(G') \leq \Delta(G) + 1$

color G' using $\Delta(G) + 1$ colors

u has at most $\Delta(G)$ nbds. in G so some color doesn't appear in his nbdhood. Color him that color. We have now colored G with $\leq \Delta(G) + 1$ colors.