Math 574, Exam 2, Spring 2006

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

There are 10 problems. Each problem is worth 5 points. SHOW your work. Make your work be coherent and clear. Write in complete sentences whenever this is possible. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will post the solutions on my website a few hours after the exam is finished.

1. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$ and let $f\colon S \to S$ be an onto function. Does f have to be one-to-one? Prove or give a counter-example.

2. Let S be the set of positive integers and let $f\colon S \to S$ be an onto function. Does f have to be one-to-one? Prove or give a counter-example.

3. Recall that the Fibonacci numbers are: $f_1 = 1$, $f_2 = 1$, and for $n \ge 3$ $f_n = f_{n-1} + f_{n-2}$. Prove that $f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$ whenever n is a positive integer.

4. Let S, T, and U be sets, and let $f\colon S \to T$ and $g\colon T \to U$ be functions. Suppose that $g \circ f$ is onto. For each question, prove or give a counterexample.
   (a) Does f have to be onto?
   (b) Does g have to be onto?

5. What is a closed formula for $\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3$? Prove your answer. (Recall that a closed formula does not have any summation signs or any dots.)

6. Goldbach's conjecture states that every even integer greater than 2 is the sum of two primes. Prove that Goldbach's conjecture is equivalent to the statement that every integer greater than 5 is the sum of three primes.

7. Prove that every integer greater than 11 is the sum of 2 composite numbers.

8. For each positive integer n, let S_n be the following set of real numbers:

$$S_n = \{x \in \mathbb{R} \mid \tfrac{1}{n} \leq x < 2 + \tfrac{1}{n}\}.$$

What is $\bigcup_{n=1}^{75} S_n$? What is $\bigcap_{n=1}^{75} S_n$? I only want the answer. I do not need to see any work.

9. Let S be a set of $n+1$ integers between 1 and $2n$. Prove that at least one integer from S divides another integer from S.

10. Prove that for every positive integer n, there does exist a set T of n integers between 1 and $2n$ such that no integer from T divides any other integer from T.