**Math 574, Exam 2, Solutions, Spring 2006**

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

There are 10 problems. Each problem is worth 5 points. SHOW your work. Make your work be coherent and clear. Write in complete sentences whenever this is possible. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will post the solutions on my website a few hours after the exam is finished.

1. **Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$ and let $f\colon S \to S$ be an onto function. Does $f$ have to be one-to-one? Prove or give a counter-example.**

OF COURSE. I am supposed to prove that if $f$ is onto, then $f$ is one-to-one. I prove the contrapositive of the given statement. That is, I prove that if $f$ is not one-to-one, then $f$ is not onto.

Suppose $f$ is not one-to-one. Then there are two elements $s \neq s'$ in $S$ with $f(s) = f(s')$. The domain of $S$ consists of $15$ elements. Two of these elements are sent to the same place. So the image of $f$ contains AT MOST $14$ elements. The target of $f$ consists of $15$ elements. The target of $f$ has at least one more element than the image of $f$. Thus, there is at least one $s'' \in S$ with $s''$ not in the image of $S$. We conclude that $f$ is not onto.

2. **Let $S$ be the set of positive integers and let $f\colon S \to S$ be an onto function. Does $f$ have to be one-to-one? Prove or give a counter-example.**

OF COURSE NOT!!! Define $f(n)$ to be the greatest integer less than or equal to $\frac{n+1}{2}$. (In other words, $f(1) = f(2) = 1$, $f(3) = f(4) = 2$, $f(5) = f(6) = 3$, etc.) We see that $f$ is onto but not one-to-one.

3. **Recall that the Fibonacci numbers are: $f_1 = 1$, $f_2 = 1$, and for $n \geq 3$ $f_n = f_{n-1} + f_{n-2}$. Prove that $f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$ whenever $n$ is a positive integer.**

We prove this result by induction on $n$.

**Base case:** When $n = 1$, the left side is $f_1 = 1$ and the right side is $f_2 = 1$. We have equality.

**Inductive Hypothesis:** Fix a positive integer $n$. Assume that

$$f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}.$$

**We will prove that:** $f_1 + f_3 + \cdots + f_{2n-1} + f_{2n+1} = f_{2n+2}$. The inductive hypothesis ensures that

$$f_1 + f_3 + \cdots + f_{2n-1} + f_{2n+1} = (f_1 + f_3 + \cdots + f_{2n-1}) + f_{2n+1} = f_{2n} + f_{2n+1}.$$

The definition of the Fibonacci numbers tells us that $f_{2n} + f_{2n+1} = f_{2n+2}$. We have completed the proof of the inductive step; and therefore we have completed the proof the result.

4. **Let $S$, $T$, and $U$ be sets, and let $f\colon S \to T$ and $g\colon T \to U$ be functions. Suppose that $g \circ f$ is onto. For each question, prove or give a counterexample.**
   **(a) Does $f$ have to be onto?**
   **(b) Does $g$ have to be onto?**

   (a) The function $f$ does NOT have to be onto. Consider $S = U = \{1\}$, $T = \{1, 2\}$, $f(1) = 1$, $g(1) = g(2) = 1$. We see that $g \circ f$ is onto, but $f$ is not onto.
   (b) The function $g$ DOES have to be onto. Let $u$ be an arbitrary element of $U$. The function $g \circ f$ is onto; so, there exists an element $s \in S$ with $g \circ f(s) = u$. Thus $f(s)$ is an element of $T$ and $g$ sends THIS element of $T$ to $u$.

5. **What is a closed formula for $\sum\limits_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3$? Prove your answer. (Recall that a closed formula does not have any summation signs or any dots.)**

We prove by induction that $\sum\limits_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$.

**Base case:** We see that when $n = 1$, then $\sum\limits_{k=1}^{n} k^3$ and $\frac{n^2(n+1)^2}{4}$ are both equal to $1$.

**Induction Hypothesis:** Fix an integer $n$ with $1 \le n$. Assume

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$

We will prove that $\sum_{k=1}^{n+1} k^3 = \frac{(n+1)^2(n+2)^2}{4}$ . The left side is equal to

$$\sum_{k=1}^{n} k^3 + (n+1)^3.$$

We apply the induction hypothesis to see that the left side is equal to

$$\frac{n^2(n+1)^2}{4} + (n+1)^3 = \frac{(n+1)^2}{4}[n^2 + 4(n+1)] = \frac{(n+1)^2}{4}[n^2 + 4n + 4]$$

$$= \frac{(n+1)^2}{4}(n+2)^2,$$

and this is the right side. We have completed the proof of the inductive step; and therefore, we have completed the proof of the result.

6. **Goldbach's conjecture states that every even integer greater than $2$ is the sum of two primes. Prove that Goldbach's conjecture is equivalent to the statement that every integer greater than $5$ is the sum of three primes.**

**Assume the original conjecture. Prove the alternate form.** Let $n$ be an integer greater than $5$. If $n$ is even, then $n-2$ is an even integer greater than $2$ and Goldbach's conjecture ensures that there exist prime numbers $p$ and $q$ with $p+q=n-2$. Thus, $p+q+2=n$ and the conclusion of the alternate form holds for $n$. If $n$ is odd, then $n-3$ is an even integer greater than $2$. Once again Goldbach's conjecture ensures that there exist prime numbers $p$ and $q$ with $p+q=n-3$. Thus, $p+q+3=n$. In any event, $n$ is the sum of three primes.

**Assume the alternate form. Prove the original conjecture.** Let $n > 2$ be an even integer. We see that $n+2$ is an arbitrary integer greater than $5$. The alternate form of the conjecture ensures that there exist prime numbers $p$, $q$, and $r$ with $n+2=p+q+r$. We notice that at least one of the numbers $p$, $q$, and $r$ must be even (because three odd numbers add up to an odd number and $n+2$ is even). The only even prime number is $2$. So one of the three prime numbers $p$, $q$ or $r$ is equal to $2$. Re-label, if necessary, in order to have $r=2$. We now subtract $2$ from each side of $n+2=p+q+2$ to see that $n=p+q$.

7. **Prove that every integer greater than $11$ is the sum of $2$ composite numbers.**

Let $n$ be an integer greater than $11$. We notice that $n-4$, $n-6$, and $n-8$ all are integers greater than $3$. Notice that $3$ must divide one of these three integers.

Indeed, there are only three possibilities for the remainder after $n-8$ is divided by $3$. If the remainder is $0$, then $3$ divides $n-8$. If the remainder is $1$, then $3$ divides $n-6$. If the remainder is $2$, then $3$ divides $n-4$. Thus, one of the three numbers $n-8$, $n-6$, and $n-4$ are composite. On the other hand, $4$, $6$, and $8$ all are composite; so, $n=(n-4)+4=(n-6)+6=(n-8)+8$ is the sum of two composite numbers.

8. **For each positive integer $n$, let $S_n$ be the following set of real numbers:**

$$S_n = \{x \in \mathbb{R} \mid \tfrac{1}{n} \le x < 2 + \tfrac{1}{n}\}.$$

**What is $\bigcup_{n=1}^{75} S_n$? What is $\bigcap_{n=1}^{75} S_n$? I only want the answer. I do not need to see any work.**

We see that

$$\boxed{\bigcup_{n=1}^{75} S_n = \{x \in \mathbb{R} \mid \tfrac{1}{75} \le x < 3 \quad \text{and} \quad \bigcap_{n=1}^{75} S_n = \{x \in \mathbb{R} \mid 1 \le x < 2 + \tfrac{1}{75}\}.}$$

9. **Let $S$ be a set of $n+1$ integers between $1$ and $2n$. Prove that at least one integer from $S$ divides another integer from $S$.**

We will prove the statement by induction on $n$.

**Base case:** If $n=1$, then $S$ consists of two numbers from $\{1,2\}$; so, $S=\{1,2\}$ and one of the integers from $S$ (namely $1$) does indeed the other integer from $S$ (namely $2$).

**Inductive step:** Let $n$ be some fixed integer with $2 \le n$. We suppose that every set $T$ of $n$ integers between $1$ and $2n-2$ contains an integer which divides another integer from the set $T$.

Let $S$ be a set of $n+1$ integers between $1$ and $2n$. **We will prove** that at least one integer from $S$ divides another integer from $S$.

We give names to the elements of $S$: $s_1 < s_2 < \cdots < s_{n+1}$. There are three cases.

**Case 1:** $s_{n+1} < 2n$. In this case, $s_n < 2n-2$ (since $s_n < s_{n+1} \le 2n-1$). We may apply the induction hypothesis to the set $T = S \setminus \{s_{n+1}\}$. We notice that $T$ is a set of $n$ integers between $1$ and $2n-2$. The induction hypothesis guarantees that some element of $T$ divides some other element of $T$. But $T$ sits inside $S$; so, some element of $S$ divides some other element of $S$.

**Case 2: $s_{n+1} = 2n$ and some $s_i$ divides $s_{n+1}$ for some $i < n+1$.** There is nothing for us to prove in this case; since, in this case, one element of $S$ (namely $s_i$) divides another element of $S$ (namely $s_{n+1}$).

**Case 3: $s_{n+1} = 2n$ and no $s_i$ divides $s_{n+1}$ for any $i < n+1$.** First notice that $n$ is not an element of $S$ because $n$ divides $2n$, but none of the elements of $S$ (except $s_{n+1}$) divide $2n$. Let $T$ be the set $\{n\} \cup S \setminus \{s_n, s_{n+1}\}$. Observe that the induction hypothesis applies to $T$. Indeed, $T$ consists of $n$ integers between 1 and $2n-2$. (We know that $s_{n-1} < s_n < s_{n+1} = 2n$.) The induction hypothesis guarantees that some element of $T$ divides some other element of $T$. We are not quite finished yet, because $T$ contains $n$, which is not in $S$. We have to make sure that the division $t_i | t_j$, for some $t_i \neq t_j$ in $T$, which is guaranteed by the induction hypothesis, involves the elements of $S \cap T$ and not $n$. But this is easy. We must rule out $n$ as the element of $t_i$ of $T$ and also as the element $t_j$ of $T$.

We make sure that $n$ does not divide any element of $T$ (other than $n$ itself): Every element that of $T$ is less than $2n$, and $n$ does not divide any integers between 1 and $2n-1$, except $n$.

We make sure that none of the elements of $T$ (other than $n$) divide $n$. Our hypothesis for case 3 says that none of the elements of $T$ (other than $n$) divide $2n$; and therefore, none of the elements of $T$ (other than $n$) divide $n$.

The proof of case 3 is now complete. The induction hypothesis guarantees that some element of $T$ divides some other element of $T$. We know that neither of these elements is $n$. Every element of $T$, other than $n$, is also in $S$; hence, some element of $S$ divides some other element of $S$.

In each of the three cases, we have proven that some element of $S$ divides some other element of $S$. The proof of the inductive step is complete; and therefore the proof of the result is complete.

10. **Prove that for every positive integer $n$, there does exist a set $T$ of $n$ integers between 1 and $2n$ such that no integer from $T$ divides any other integer from $T$.**

Let $\boxed{T = \{n+1, n+2, \ldots, 2n\}}$. We see that $T$ consists of $n$ distinct integers between 1 and $2n$; however, no element of $T$ divides any other element of $T$.