(6) How many pure monomials of degree d can be formed using the variables $x_1, x_2, \ldots, x_n$? (The expression $x_1^{e_1} x_2^{e_2} \cdots x_n^{e_n}$ is a pure monomial of degree d if each e_i is a non-negative integer and $e_1 + \cdots + e_n = d$. For example, the pure monomials of degree 3 using the variables x_1 and x_2 are $x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^3$.)

This is a "combinations with replacement" problem. Imagine n bins. Bin 1 has x_1. Bin 2 has x_2, etc. We identify each pure monomial of degree d in n variables with a word made out of d p's and n-1 s's. There $\binom{n-1+d}{d}$ such words.

(7) Prove that $\sum_{k=0}^{n} k^3 = \frac{n^2(n+1)^2}{4}$.

Proof by induction on n.

The Formula is true for n=1 since $1^3 = \frac{1^2 2^2}{4}$.

Induction Hypothesis: Assume $\sum_{k=0}^{n} k^3 = \frac{n^2(n+1)^2}{4}$.

We must show $\sum_{k=0}^{n+1} k^3 = \frac{(n+1)^2(n+2)^2}{4}$

$$\text{L.H.S.} = \sum_{k=0}^{n} k^3 + (n+1)^3 \underset{\text{I.H.}}{=} \frac{n^2(n+1)^2}{4} + (n+1)^3 = \frac{(n+1)^2}{4}\left[n^2 + 4n + 4\right]$$

$$= \frac{(n+1)^2}{4}(n+2)^2 = \text{R.H.S.}$$