Math 574, Exam 1, Spring 2006

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

There are 10 problems. Each problem is worth 5 points. SHOW your work. Make your work be coherent and clear. Write in complete sentences whenever this is possible. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will post the solutions on my website a few hours after the exam is finished.

1. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$ and let $f\colon S \to S$ be a one-to-one function. Does f have to be onto? Prove or give a counter-example.

2. Let S be the set of positive integers and let $f\colon S \to S$ be a one-to-one function. Does f have to be onto? Prove or give a counter-example.

3. Let A and B be sets. (Recall that $A \setminus B = \{a \in A \mid a \notin B\}$.) Simplify $A \setminus (A \setminus B)$. Prove your answer.

4. Let f be a function from the real numbers to the real numbers, and let a be a real number. What is the negation of the statement: "For all real numbers $\varepsilon > 0$, there exists a real number $\delta > 0$, such that if x is a real number, with $0 < |x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$"?

5. Goldbach's conjecture states that every even integer greater than 2 is the sum of two primes. Prove that Goldbach's conjecture is equivalent to the statement that every integer greater than 5 is the sum of three primes.

6. Prove that the square of an integer not divisible by 5 leaves a remainder of 1 or 4 when divided by 5.

7. For each positive integer n, let S_n be the following set of real numbers:

$$S_n = \{x \in \mathbb{R} \mid \tfrac{-1}{n} < x < 2 + \tfrac{1}{n}\}.$$

What is $\bigcup_{n=1}^{\infty} S_n$? What is $\bigcap_{n=1}^{\infty} S_n$? I only want the answer. I do not need to see any work.

8. Let $A = \{t, u, v, w\}$ and let S_1 be the set of all subsets of A that do not contain w and S_2 the set of all subsets of A that do contain w.
   (a) List the elements of S_1.
   (b) List the elements of S_2.

9. Determine the truth value of the following statements. Explain.
   (a) $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$ with $x^2 = y$.
   b) $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$ with $x = y^2$.

10. Consider the statement "if $3 < x$, then $9 < x^2$".
   (a) What is the converse of the original statement?
   (b) Is (a) logically equivalent to the original statement?
   (c) What is the contrapositive of the original statement?
   (d) Is (c) logically equivalent to the original statement?