Math 574, Exam 1, Spring 2006

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

There are 10 problems. Each problem is worth 5 points. SHOW your work. Make your work be coherent and clear. Write in complete sentences whenever this is possible. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will post the solutions on my website a few hours after the exam is finished.

1. **Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$ and let $f\colon S \to S$ be a one-to-one function. Does f have to be onto? Prove or give a counter-example.**

The domain S has 15 elements. The function f is one-to-one, so there are 15 distinct elements in the image of f. On the other hand, the target S only has 15 elements, so every element of S is in the image of f; in other words, $\boxed{\text{Yes, } f \text{ is onto.}}$

2. **Let S be the set of positive integers and let $f\colon S \to S$ be a one-to-one function. Does f have to be onto? Prove or give a counter-example.**

$\boxed{\text{No}}$ Consider the function $f\colon S \to S$, which is given by $f(n) = n + 1$. We see that f is one-to-one, but $1 \in S$ and there does not exist a positive integer n with $f(n) = 1$.

3. **Let A and B be sets. (Recall that $A \setminus B = \{a \in A \mid a \notin B\}$.) Simplify $A \setminus (A \setminus B)$. Prove your answer.**

The set $A \setminus (A \setminus B)$ is equal to $\boxed{A \cap B}$.

$\underline{A \cap B \subseteq A \setminus (A \setminus B)}$:
Take $x \in A \cap B$. So $x \in A$ and $x \notin A \setminus B$. It follows that $x \in A \setminus (A \setminus B)$.

$\underline{A \setminus (A \setminus B) \subseteq A \cap B}$:
Take $x \in A \setminus (A \setminus B)$. This means that $x \in A$, but $x \notin A \setminus B$. The only way for x to be in A but not in $A \setminus B$ is for x to be in B. Thus, $x \in A \cap B$.

4. **Let f be a function from the real numbers to the real numbers, and let a be a real number. What is the negation of the statement: "For all real numbers $\varepsilon > 0$, there exists a real number $\delta > 0$, such that if x is a real number, with $0 < |x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$"?**

There exists a real number $\varepsilon > 0$, such that for all real numbers $\delta > 0$, there exists a real number x with $0 < |x - a| < \delta$ but $|f(x) - f(a)| \geq \varepsilon$.

5. **Goldbach's conjecture states that every even integer greater than 2 is the sum of two primes. Prove that Goldbach's conjecture is equivalent to the statement that every integer greater than 5 is the sum of three primes.**

Assume the original conjecture. Prove the alternate form. Let n be an integer greater than 5. If n is even, then $n - 2$ is an even integer greater than 2 and Goldbach's conjecture ensures that there exist prime numbers p and q with $p + q = n - 2$. Thus, $p + q + 2 = n$ and the conclusion of the alternate form holds for n. If n is odd, then $n - 3$ is an even integer greater than 2. Once again Goldbach's conjecture ensures that there exist prime numbers p and q with $p + q = n - 3$. Thus, $p + q + 3 = n$. In any event, n is the sum of three primes.

Assume the alternate form. Prove the original conjecture. Let $n > 2$ be an even integer. We see that $n + 2$ is an arbitrary integer greater than 5. The alternate form of the conjecture ensures that there exist prime numbers p, q, and r with $n + 2 = p + q + r$. We notice that at least one of the numbers p, q, and r must be even (because three odd numbers add up to an odd number and $n + 2$ is even). The only even prime number is 2. So one of the three prime numbers p, q or r is equal to 2. Re-label, if necessary, in order to have $r = 2$. We now subtract 2 from each side of $n + 2 = p + q + 2$ to see that $n = p + q$.

6. **Prove that the square of an integer not divisible by 5 leaves a remainder of 1 or 4 when divided by 5.**

Let n be an arbitrary integer not divisible by 5. There are four cases.

Case 1: If $n = 5k + 1$ for some integer k, then $n^2 = 25k^2 + 10k + 1 = 5(5k^2 + 2k) + 1$, which has remainder 1 when divided by 5.

Case 2: If $n = 5k + 2$ for some integer k, then $n^2 = 25k^2 + 20k + 4 = 5(5k^2 + 4k) + 4$, which has remainder 4 when divided by 5.

Case 3: If $n = 5k + 3$ for some integer k, then $n^2 = 25k^2 + 30k + 9 = 5(5k^2 + 6k + 1) + 4$, which has remainder 4 when divided by 5.

Case 4: If $n = 5k + 4$ for some integer k, then $n^2 = 25k^2 + 40k + 16 = 5(5k^2 + 8k + 3) + 1$, which has remainder 1 when divided by 5.

7. **For each positive integer n, let S_n be the following set of real numbers:**

$$S_n = \{x \in \mathbb{R} \mid \frac{-1}{n} < x < 2 + \frac{1}{n}\}.$$

What is $\bigcup_{n=1}^{\infty} S_n$? What is $\bigcap_{n=1}^{\infty} S_n$? I only want the answer. I do not need to see any work.

We see that $\bigcup_{n=1}^{\infty} S_n = S_1 = \{x \in \mathbb{R} \mid -1 < x < 3\}$, and that

$$\bigcap_{n=1}^{\infty} S_n = \{x \in \mathbb{R} \mid 0 \leq x \leq 2\}.$$

8. **Let $A = \{t, u, v, w\}$ and let S_1 be the set of all subsets of A that do not contain w and S_2 the set of all subsets of A that do contain w.**
(a) List the elements of S_1.
(b) List the elements of S_2.

(a) The elements of S_1 are: $\emptyset$, $\{t\}$, $\{u\}$, $\{v\}$, $\{t, u\}$, $\{t, v\}$, $\{u, v\}$, $\{t, u, v\}$.
(b) The elements of S_2 are: $\{w\}$, $\{t, w\}$, $\{u, w\}$, $\{v, w\}$, $\{t, u, w\}$, $\{t, v, w\}$, $\{u, v, w\}$, $\{t, u, v, w\}$.

9. **Determine the truth value of the following statements. Explain.**
(a) $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$ **with** $x^2 = y$.
b) $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$ **with** $x = y^2$.

(a) TRUE. Once x has been chosen, then just take $y = x^2$. Observe that y is a real number.
(b) FALSE. Take $x = -1$. We see that x does not have a square root in $\mathbb{R}$.

10. **Consider the statement "if $3 < x$, then $9 < x^2$".**
(a) What is the converse of the original statement?
(b) Is (a) logically equivalent to the original statement?
(c) What is the contrapositive of the original statement?
(d) Is (c) logically equivalent to the original statement?

(a) The converse of the original statement is "if $9 < x^2$, then $3 < x$".
(b) NO.
(c) The contrapositive of the original statement is "if $9 \geq x^2$, then $3 \geq x$.
(d) YES.