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Quiz for June 15, 2004

Let $\{a_n\}$ and $\{b_n\}$ be Cauchy sequences. Prove that the sequence $\{a_nb_n\}$ is also a Cauchy sequence.

ANSWER: Let $\varepsilon > 0$ be arbitrary, but fixed.

• The sequence $\{a_n\}$ is a Cauchy sequence; so, there exists n_1 such that whenever $n_1 \le n, m$, then $|a_n - a_m| \le 1$. In particular, whenever $n_1 \le n$, then

$$|a_n| - |a_{n_1}| \le ||a_n| - |a_{n_1}|| \le |a_n - a_{n_1}| \le 1;$$

so,

(1) $$|a_n| \le |a_{n_1}| + 1.$$

• The exact same reasoning produces an integer n_2 with the property that if $n_2 \le m$, then

(2) $$|b_m| \le |b_{n_2}| + 1.$$

• The sequence $\{a_n\}$ is a Cauchy sequence; so, there exists n_3 such that whenever $n_3 \le n, m$, then

(3) $$|a_n - a_m| \le \frac{\varepsilon}{2(|b_{n_2}| + 1)}.$$

• The sequence $\{b_n\}$ is a Cauchy sequence; so, there exists n_4 such that whenever $n_4 \le n, m$, then

(4) $$|b_n - b_m| \le \frac{\varepsilon}{2(|a_{n_1}| + 1)}.$$

Pick n_0 to be the maximum of n_1, n_2, n_3, and n_4. If $n_0 \le n, m$, then the triangle inequality and (1), (2), (3), and (4) give:

$$|a_nb_n - a_mb_m| = |a_nb_n - a_nb_m + a_nb_m - a_mb_m| \le |a_nb_n - a_nb_m| + |a_nb_m - a_mb_m|$$

$$= |a_n||b_n - b_m| + |a_n - a_m||b_m| \le (|a_{n_1}| + 1)\tfrac{\varepsilon}{2(|a_{n_1}|+1)} + \tfrac{\varepsilon}{2(|b_{n_2}|+1)}(|b_{n_2}| + 1) = \varepsilon.$$