Math 554, Final Exam, Summer 2006

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. **Leave room on the upper left hand corner of each page for the staple.** Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc; although, by using enough paper, you can do the problems in any order that suits you.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**. Otherwise, get your grade from VIP.

There are 11 problems. The exam is worth a total of 100 points.

I will post the solutions on my website later this afternoon.

Record ALL of your answers in complete sentences.

1. (9 points) Define "continuous". **Use complete sentences.** Include everything that is necessary, but nothing more.

2. (9 points) Define "supremum". **Use complete sentences.** Include everything that is necessary, but nothing more.

3. (9 points) PROVE that the continuous image of a compact set is compact.

4. (9 points) STATE and PROVE the Nested Interval Property.

5. (10 points) Let A be a set. For each $\alpha \in A$, let U_α be an open subset of $\mathbb{R}$ and F_α be a closed subset of $\mathbb{R}$. For each question: if the answer is yes, then PROVE the assertion; if the answer is no, then give a counter example.
   (a) Does $\bigcup_{\alpha \in A} U_\alpha$ have to be open?
   (b) Does $\bigcap_{\alpha \in A} U_\alpha$ have to be open?
   (c) Does $\bigcup_{\alpha \in A} F_\alpha$ have to be closed?
   (d) Does $\bigcap_{\alpha \in A} F_\alpha$ have to be closed?

6. (9 points) Let $E = \{1 - \frac{1}{n} \mid n \in \mathbb{N}\}$ and let $F = E \cup \{1\}$.
   (a) Give an example of an open cover of E which does not admit a finite subcover. PROVE all of your assertions.
   (b) Prove DIRECTLY (that is, do not quote any Theorems) that every open cover of F does admit a finite subcover.

7. (9 points) Consider the sequence $\{a_n\}$ with $a_n = \sum_{k=1}^{n} \frac{1}{k!}$. Prove that $\{a_n\}$ is a Cauchy sequence.

8. (9 points) Let $a_1 \neq a_2$ be real numbers. For $n \geq 3$, let $a_n = \frac{3}{4}a_{n-1} + \frac{1}{4}a_{n-2}$. PROVE that the sequence $\{a_n\}$ is a contractive sequence.

9. (9 points) Let $\{a_n\}$ be a sequence of positive real numbers. Suppose that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$ for some real number L with $L < 1$. Does the sequence $\{a_n\}$ have to converge? If the answer is yes, then PROVE the assertion; if the answer is no, then give a counter example.

10. (9 points) Let A and B be non-empty sets, and let $f\colon A \to B$ and $g\colon B \to C$ be functions. Suppose that the function $g \circ f$ is onto. For each question: if the answer is yes, then PROVE the assertion; if the answer is no, then give a counter example.
    (a) Does f have to be onto?
    (b) Does g have to be onto?

11. (9 points) Let A and B be nonempty subsets of positive real numbers that are bounded from above. Let $C = \{ab \mid a \in A \text{ and } b \in B\}$. PROVE that $\sup C = (\sup A)(\sup B)$.