Math 554, Exam 4, Summer 2005 Solution

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc; although, by using enough paper, you can do the problems in any order that suits you.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

There are 7 problems. Problem 1 is worth 8 points. Problems 2 through 7 are worth 7 points each. The exam is worth a total of 50 points.

If you would like, I will leave your graded exam outside my office door. You may pick it up any time before the next class. **If you are interested, be sure to tell me.**

I will post the solutions on my website shortly after the class is finished.

1. **Let $E = \{q \in \mathbb{Q} \mid 0 \le q \le 1\}$. Is E a compact set? Explain thoroughly. (Recall that $\mathbb{Q}$ is the set of rational numbers.)**

The set E is NOT compact. We proved that a set is compact if and only if it is closed and bounded. The set E is NOT closed, therefore, the set E is NOT compact. We know that E is not closed because every irrational number in $[0,1]$ is a limit point of E, but is not in E. Indeed, if r is an irrational number with $r \in [0,1]$, then every ε neighborhood of r contains elements of E because there exists a rational number between any two real numbers.

2. **Define *compact*. Use complete sentences. Include everything that is necessary, but nothing more.)**

The subset K of $\mathbb{R}$ is *compact* if every open cover of K admits a finite subcover.

3. **Define *continuous*. Use complete sentences. Include everything that is necessary, but nothing more.**

Let E be a subset of $\mathbb{R}$. The function $f\colon E \to \mathbb{R}$ is continuous at the point p of E, if, for all $\varepsilon > 0$, there exists $\delta > 0$, such that whenever $|x - p| < \delta$ and $x \in E$, then $|f(x) - f(p)| < \varepsilon$.

4. **Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined by**

$$f(x) = \begin{cases} x^3 & \textbf{if } x \textbf{ is rational} \\ 5x - 2 & \textbf{if } x \textbf{ is irrational.} \end{cases}$$

Give a complete "ε-δ" proof that f is continuous at $p = 2$.

We see that $f(2) = 8$. Let $\varepsilon > 0$ be fixed, but arbitrary. We must prove that there exists $\delta > 0$ such that whenever $|x-2| < \delta$, then $|f(x) - 8| < \varepsilon$.

We know the formula for the difference of perfect cubes. (Or we do a long division.) At any rate, we see that

$$|x^3 - 8| = |(x-2)(x^2 + 2x + 4)| \le |x-2|(|x|^2 + 2|x| + 4).$$

We can make $|x-2|$ be as small as we might like. We "are only interested" in x's near 2, so we can keep $(|x|^2 + 2|x| + 4)$ from being too big. Said with more details: if $|x-2| < 1$, then $|x| < 3$ and $|x|^2 + 2|x| + 4 < 9 + 6 + 4 = 19$. It follows that

$$(1) \qquad |x-2| < \min\{1, \tfrac{\varepsilon}{19}\} \implies |x^3 - 8| \le |x-2|(|x|^2 + 2|x| + 4) < \tfrac{\varepsilon}{19} 19 = \varepsilon.$$

We also see that

$$|(5x-2) - 8| = |5x - 10| = 5|x-2|.$$

So,

$$(2) \qquad |x-2| < \tfrac{\varepsilon}{5} \implies |(5x-2) - 8| = 5|x-2| < 5\tfrac{\varepsilon}{5} = \varepsilon.$$

We know that $f(x)$ is equal to either x^3 or $5x - 2$. Let $\delta = \min\{1, \frac{\varepsilon}{19}, \frac{\varepsilon}{5}\}$. Combine (1) and (2) to see that if $|x-2| < \delta$, then $|f(x) - 8)| < \varepsilon$.

5. **STATE the theorem which relates the limit of a function and the limit of various sequences. Use complete sentences. Include everything that is necessary, but nothing more.**

Let E be a subset of $\mathbb{R}$, p be a limit point of E, $f\colon E \to \mathbb{R}$ be a function, and L be a real number. The following statments are equivalent.

(a) The limit $\lim\limits_{x \to p} f(x)$ is equal to L.

(b) For every sequence $\{x_n\}$ in E which converges to p with $x_n \ne p$ for all n, the sequence $\{f(x_n)\}$ converges to L.

6. **Let K be a compact set and let $f\colon K \to \mathbb{R}$ be a continuous function. Prove that the image $f(K)$ is a compact set. (I want to see a complete proof.)**

Let $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ be an open cover of $f(K)$. For each point $p \in K$, the element $f(p)$ is in $f(K)$. The set $\mathcal{U}$ covers $f(K)$, so there is an index α_p in A, such that $f(p)$ is in U_{α_p}. The function f is continuous at p; so there exists a $\delta_p > 0$ such that $f(N_{\delta_p}(p) \cap K) \subseteq U_{\alpha_p}$. We create such a neighborhood $N_{\delta_p}(p)$ for each $p \in K$. We see that $\mathcal{N} = \{N_{\delta_p}(p) \mid p \in K\}$ is an open cover of K. The set K is compact; consequently, there exist $p_1, \dots, p_n$ in K such that the set $\{N_{\delta_{p_1}}(p_1), \dots, N_{\delta_{p_n}}(p_n)\}$ covers K. It follows that $\{f(N_{\delta_{p_1}}(p_1) \cap K), \dots, f(N_{\delta_{p_n}}(p_n) \cap K)\}$ covers $f(K)$. But $f(N_{\delta_{p_i}}(p_i) \cap K) \subseteq U_{\alpha_{p_i}}$, for all i; therefore, $\{U_{\alpha_{p_1}}, \dots, U_{\alpha_{p_n}}\}$ covers $f(K)$.

7. **Let f be a continuous function from the closed interval $[a,b]$ to $\mathbb{R}$. Let $\varepsilon > 0$ be fixed. Prove that there exists $\delta > 0$ such that: whenever x and y are in $[a,b]$ with $|x-y| < \delta$, then $|f(x)-f(y)| < \varepsilon$. (Notice that you are supposed to prove that one δ works for every x and y.)**

The function f is continuous on $[a,b]$, so for each point $p \in [a,b]$ there exists "a δ depending on p", call it $\delta_p > 0$, such that whenever

$$x \in [a,b] \text{ with } |x-p| < \delta_p \implies |f(x)-f(p)| < \tfrac{\varepsilon}{2}. \tag{3}$$

Let $\mathcal{N} = \{N_{\frac{\delta_p}{2}}(p) \mid p \in [a,b]\}$. We see that $\mathcal{N}$ is an open cover of the compact set $[a,b]$. Thus, there exists a finite set of points $p_1, \dots, p_\ell$ in $[a,b]$ so that $\{N_{\frac{\delta_{p_1}}{2}}(p_1), \dots, N_{\frac{\delta_{p_\ell}}{2}}(p_\ell)\}$ covers $[a,b]$. Let $\delta = \min\{\frac{\delta_{p_1}}{2}, \dots \frac{\delta_{p_\ell}}{2}\}$. The number δ is the minimum of a FINITE set of POSITIVE numbers, so $\delta > 0$. I claim that this one δ works for ALL x and y. Suppose, x, y are in $[a,b]$ with $|x-y| < \delta$. The set $\{N_{\frac{\delta_{p_1}}{2}}(p_1), \dots, N_{\frac{\delta_{p_\ell}}{2}}(p_\ell)\}$ covers $[a,b]$, so there is an index i, with $1 \le i \le \ell$ with $|x-p_i| < \frac{\delta_{p_i}}{2}$. It follows that

$$|y-p_i| = |y-x+x-p_i| \le |y-x| + |x-p_i| \le \delta + \frac{\delta_{p_i}}{2} \le \frac{\delta_{p_i}}{2} + \frac{\delta_{p_i}}{2} = \delta_{p_i}.$$

Use (3) twice at $p = p_i$ to see

$$|f(x)-f(y)| = |f(x)-f(p_i)+f(p_i)-f(y)| \le |f(x)-f(p_i)| + |f(p_i)-f(y)|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$