Math 554, Final Exam Summer 2004

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Take enough space for each problem. Turn in your solutions in the order: problem 1, problem 2, ... ; although, by using enough paper, you can do the problems in any order that suits you.

There are 16 problems. Problems 1, 2, 3, and 4 are worth 7 points each. Problems 5 through 16 are worth 6 points each. The exam is worth a total of 100 points.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**. Otherwise, get your course grade from VIP.

I will post the solutions on my website shortly after the class is finished.

1. Define *Cauchy sequence*. Use complete sentences.

2. Let $f\colon E \to \mathbb{R}$ be a function which is defined on a subset E of $\mathbb{R}$. Define $\lim_{x\to p} f(x) = L$. Use complete sentences. (Be sure to tell me what kind of a thing p is, and what kind of a thing L is.)

3. Define *continuous*. Use complete sentences.

4. STATE either version of the Bolzano-Weierstrass Theorem.

5. PROVE either version of the Bolzano-Weierstrass Theorem.

6. PROVE that every Cauchy sequence converges.

7. Let E be a set which is not closed. PROVE that E is not compact by constructing an open cover of E which does not admit a finite subcover.

8. PROVE that the continuous image of a compact set is compact.

9. Let I be an interval and $f\colon I \to \mathbb{R}$ be a function which is differentiable at the point p of I. PROVE that f is continuous at p.

10. Let A and B be non-empty sets, and let $f\colon A \to B$ and $g\colon B \to A$ be functions. Suppose that $g \circ f$ is the identity function on A. (In other words, $g(f(a)) = a$ for all a in A.) Does f have to be onto? If yes, PROVE the result. If no, then give an EXAMPLE.

11. Give an example of a countable set E and an open cover $\mathcal{U}$ of E which does not admit a finite subcover of E.

12. Let $f(x) = \begin{cases} x^3 & \text{if } x \text{ is irrational} \\ 0 & \text{if } x \text{ is rational.} \end{cases}$ Does $f'(0)$ exist? PROVE your answer completely, using ε's and δ's.

13. Let $f(x) = \begin{cases} 5x-3 & \text{if } x \leq 1 \\ 4-2x & \text{if } 1 < x. \end{cases}$ Is f continuous at $x = 1$? PROVE your answer completely, using ε's and δ's.

14. For each integer n, let I_n be the open interval $(\frac{1}{n}, 2 + \frac{1}{n})$. Compute $\bigcap_{n=1}^{\infty} I_n$.

15. Let $a_1 \neq a_2$ be real numbers. For $n \geq 3$, let $a_n = \frac{2}{3}a_{n-1} + \frac{1}{3}a_{n-2}$. PROVE that the sequence $\{a_n\}$ is a contractive sequence.

16. Let $a_1 = \sqrt{2}$ and for each integer $n \geq 1$, let $a_{n+1} = \sqrt{2 + a_n}$. PROVE that $a_n \leq 2$ for all n. PROVE that the sequence $\{a_n\}$ is a monotone increasing sequence.