Math 554, Exam 4, Summer 2004

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Take enough space for each problem. Turn in your solutions in the order: problem 1, problem 2, ... ; although, by using enough paper, you can do the problems in any order that suits you.

There are 8 problems. Problems 1 and 2 are worth 7 points each. Problems 3 through 8 are worth 6 points each. The exam is worth a total of 50 points.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will leave your exam outside my office door by noon tomorrow, you may pick it up any time between then and the next class.

I will post the solutions on my website shortly after the class is finished.

1. **For each natural number n, let C_n be a closed set in $\mathbb{R}$. Is the intersection $\bigcap_{n=1}^{\infty} C_n$ always a closed set? If yes, prove the result. If no, give a counterexample.**

YES! We show that the complement $\mathbb{R} \setminus \bigcap_{n=1}^{\infty} C_n$ is an open set. It is well-known (and easy to check) that

$$\mathbb{R} \setminus \bigcap_{n=1}^{\infty} C_n = \bigcup_{n=1}^{\infty} (\mathbb{R} \setminus C_n).$$

Each set C_n is closed; hence, each $\mathbb{R} \setminus C_n$ is an open set. The union of open sets is open as we saw in problem 4 on Exam 3:

For each natural number n, let U_n be an open set in $\mathbb{R}$. Is the union $\bigcup_{n=1}^{\infty} U_n$ always an open set? If yes, prove the result. If no, give a counterexample.

YES! If $p \in \bigcup_{n=1}^{\infty} U_n$, then $p \in U_{n_1}$ for some fixed n_1; hence, there exists ε such that $N_\varepsilon(p) \subseteq U_{n_1}$. It follows that $N_\varepsilon(p) \subseteq \bigcup_{n=1}^{\infty} U_n$; and therefore, $\bigcup_{n=1}^{\infty} U_n$ is an open subset of $\mathbb{R}$.

2. **For each natural number n, let C_n be a closed set in $\mathbb{R}$. Is the union $\bigcup_{n=1}^{\infty} C_n$ always a closed set? If yes, prove the result. If no, give a counterexample.**

NO! For each natural number n, let C_n be the closed interval $[\frac{1}{n}, 2 - \frac{1}{n}]$. It is easy to see that $\bigcup_{n=1}^{\infty} C_n = (0, 2)$. It is clear that the open interval $(0, 2)$ is not a closed set because this set does not contain the limit point 0.

3. **Define** *open set*. **Use complete sentences.**

The subset E of $\mathbb{R}$ is an *open set* if for all points $p \in E$, there exists an $\varepsilon > 0$ with $N_\varepsilon(p) \subseteq E$.

4. **Define** *compact*. **Use complete sentences.**

The subset K of $\mathbb{R}$ is *compact* if every open cover of K admits a finite subcover.

5. **State the Heine-Borel Theorem.**

The closed interval $[a, b]$ is compact.

6. **Prove the Heine-Borel Theorem.**

Let $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ be an open cover of $[a, b]$. Let

$$E = \{r \in [a, b] \mid [a, r] \text{ can be covered by a finite subset of } \mathcal{U}\}.$$

The set E is bounded (because every element of E is between a and b) and the set E is non-empty because $a \in E$. The least upper bound axiom assures us that $c = \sup E$ exists.

We first claim that $c \in E$. The number $c \in [a, b]$; and $[a, b]$ is covered by $\mathcal{U}$; so there is a set U_{α_0} from $\mathcal{U}$ (for some $\alpha_0 \in A$) with $c \in U_{\alpha_0}$. The set U_{α_0} is open; so some neighborhood, $N_\varepsilon(c)$, of c, is contained in U_{α_0}, for some $\varepsilon > 0$. The number c is the supremum of E, so there is an element $r \in E \cap N_\varepsilon(c)$. The interval $[a, r]$ may be covered by a finite subset set $U_{\alpha_1}, \dots, U_{\alpha_n}$ of $\mathcal{U}$, for some $\alpha_1, \dots, \alpha_n$ from A. So, $U_{\alpha_0}, U_{\alpha_1}, \dots, U_{\alpha_n}$ covers $[a, c]$; and therefore, $c \in E$.

Now, we claim that $c = b$. (Once we establish this claim, then the proof is complete.) It is clear that $c \le b$ (because b is an upper bound for E). We complete the proof of this claim by contradiction. Suppose

$$c < b. \tag{*}$$

We know from the first claim that there exists a finite subset $U_{\beta_1}, \dots, U_{\beta_m}$ of $\mathcal{U}$, (for some $\beta_1, \dots, \beta_m$ in A), which covers $[a, c]$. In particular, c is in U_{β_i} for some i, with $1 \le i \le m$. The set U_{β_i} is open; so some neighborhood $N_{\varepsilon_1}(c)$ of c, for some $\varepsilon_1 > 0$, is contained in U_{β_i}. There are numbers in $N_{\varepsilon_1}(c)$ which fall between c and b. For the sake of concreteness, there is a number d with $c < d < c + \varepsilon_1$ and $d < b$. It is clear that $U_{\beta_1}, \dots, U_{\beta_m}$ covers $[a, d]$. Thus, $d \in E$ and $\sup E = c < d$. This is impossible. Our supposition is false; c is not less than b. The only remaining option is that c is equal to b.

7. **Let** $f(x) = \begin{cases} 2x - 1 & \textbf{if } x \le 2 \\ 2x + 1 & \textbf{if } 2 < x. \end{cases}$ **What is** $\lim\limits_{x \to 2} f(x)$**? Prove your answer.**

$\boxed{\text{This limit does not exist.}}$ We proved that if $\lim\limits_{x \to 2} f(x) = L$ and $\{x_n\}$ is a sequence of real numbers, which never equals 2, but which converges to 2, then the sequence $\{f(x_n)\}$ converges to L. The sequence $\{2-\frac{1}{n}\}$ converges to 2 and never equals 2 and the sequence

$$\{f(2-\tfrac{1}{n})\} = \{2(2-\tfrac{1}{n})-1\} = \{3-\tfrac{2}{n}\}.$$

This sequence converges to 3. On the other hand, The sequence $\{2+\frac{1}{n}\}$ converges to 2 and never equals 2 and the sequence

$$\{f(2+\tfrac{1}{n})\} = \{2(2+\tfrac{1}{n})+1\} = \{5+\tfrac{2}{n}\}.$$

This sequence converges to 5. If $\lim\limits_{x\to 2} f(x)$ existed, this this limit would have to equal 3. It would also have to equal 5. Well, it is not possible for some number to equal both 3 and 5. We conclude that $\lim\limits_{x\to 2} f(x)$ does not exist.

8. **Let** $f(x) = \begin{cases} 2x-1 & \textbf{if } x \le 2 \\ 2x+1 & \textbf{if } 2 < x. \end{cases}$ **What is** $\lim\limits_{x\to 3} f(x)$**? Prove your answer.**

The limit is 7. Let $\varepsilon > 0$ be fixed, but arbitrary. Let $\delta = \min\{1, \frac{\varepsilon}{2}\}$. If $|x-3| < \delta$, then $2 < x$ and

$$|f(x)-7| = |2x+1-7| = 2|x-3| < 2\tfrac{\varepsilon}{2} = \varepsilon.$$