Math 550, Exam 2, Solution, Spring 2013

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: turn the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible.

No Calculators or Cell phones.

The solutions will be posted later today.

1. (10 points) **Let** D **be the region** $D = \{(x,y) \mid x^2+y^2 \le 1\}$ **in the** xy**-plane. Does**

$$\int\int_D \frac{\sin^2(x^2)}{\sqrt{1-x^2-y^2}}dA$$

exist? Explain thoroughly.

The function becomes infinite as one moves toward the boundary. The function is positive on D. So the Fubini Theorem applies. Either the given integral goes to infinity or else the given integral is bounded in which case the given integral does converge to some real number. If we can identify a function f with $\frac{\sin^2(x^2)}{\sqrt{1-x^2-y^2}} < f$ and $\int\int_D f dA$ converges, then the given integral also converges. (Similarly, if we were to find a function g so that $g < \frac{\sin^2(x^2)}{\sqrt{1-x^2-y^2}}$ and $\int\int_D g dA$ diverges, then the given integral would also diverge. It turns out that this does not happen in the present problem.) Consider $f = \frac{1}{\sqrt{1-x^2-y^2}}$. It is clear that $\frac{\sin^2(x^2)}{\sqrt{1-x^2-y^2}} < f$. We compute

$$\int\int_D f d = \int\int_D \frac{1}{\sqrt{1-x^2-y^2}}dA = \lim_{b\to 1^-}\int_0^b\int_0^{2\pi}\frac{r}{\sqrt{1-r^2}}d\theta dr$$

$$= 2\pi \lim_{b\to 1^-} -\sqrt{1-r^2}\Big|_0^b = 2\pi \lim_{b\to 1^-}(1-\sqrt{1-b^2}) = 2\pi < \infty.$$

$$\boxed{\text{We conclude that } \int\int_D \frac{\sin^2(x^2)}{\sqrt{1-x^2-y^2}}dA \text{ exists.}}$$

2. (10 points) **Let T be the region inside an equilateral triangle in the xy-plane. Two of the vertices of T are $(3,0)$ and $(5,0)$. Rotate T about the y-axis. What is the volume of the resulting solid? Explain thoroughly.**

Let S be the name of the resulting solid. We proved, as a homework problem, that the volume of S is equal to 2π times the area of T times the distance from the center of mass of T to the axis of revolution. (This is actually quite easy to prove!) The center of mass of T has x-coordinate equal to 4 by the symmetry of T; hence the distance from the center of mass to the axis of revolution is 4. The area of T is $1/2$ the base times the height. The base of T has length 2. To figure out the height, chop T in half and look at the right triangle with angles 30°, 60°, and 90°, base 1, and hypotenuse 2. Now it is clear that the height of T is $\sqrt{3}$. The area of T is $(1/2)2\sqrt{3}$. The volume of S is $\boxed{2\pi(\sqrt{3})4}$.

3. (10 points) **Compute $\int\int_D x dA$, where D is the region in the xy-plane bounded by the line segment joining $(2,3)$ to $(4,4)$, the line segment joining $(4,4)$ to $(5,6)$, the line segment joining $(5,6)$ to $(3,5)$, and the line segment joining $(3,5)$ to $(2,3)$. Explain thoroughly.**

First observe that D is a parallelogram. I prefer to view the vertices as column vectors:

$$v_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad v_2 = v_1 + \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad v_3 = v_1 + \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad v_4 = v_1 + \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

I am inclined to transform the unit square U onto D. I do this in two steps. First I use

$$\begin{bmatrix} u \\ v \end{bmatrix} \mapsto \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

to transform U onto the parallelogram with vertices

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

then I translate $\mathbb{R}^2$ by adding v_1 to each vector:

$$\begin{bmatrix} x \\ y \end{bmatrix} = T\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2u+v+2 \\ u+2v+3 \end{bmatrix}.$$

(Do notice that T really does carry the unit square onto D in a one-to-one manner.) Thus,

$$\int\int_D x dA = \int_0^1 \int_0^1 (2u+v+2) \cdot \text{Jac}\, du\, dv = 3 \int_0^1 \int_0^1 (2u+v+2) du\, dv$$

$$= 3 \int_0^1 (u^2 + vu + 2u)\Big|_0^1 dv = 3 \int_0^1 (3+v) dv = \boxed{3(\tfrac{7}{2})}.$$

4. (10 points) **Compute** $\int\int_D x^2 dA$**, where** D **is the region in the** xy**-plane determined by the two conditions** $0 \leq x \leq y$ **and** $x^2 + y^2 \leq 1$**. Explain thoroughly.**

We use polar coordinates. The given region is everything inside the unit circle in the first quadrant which is ABOVE the line $y = x$. The integral is equal to

$$\int_{\pi/4}^{\pi/2} \int_0^1 r^3 \cos^2 \theta \, dr \, d\theta = \frac{1}{4}\frac{1}{2} \int_{\pi/4}^{\pi/2} (1 + \cos 2\theta) \, d\theta = \frac{1}{8} \left(\theta + \frac{\sin(2\theta)}{2} \right) \Big|_{\pi/4}^{\pi/2} = \boxed{\frac{1}{8} \left(\frac{\pi}{4} - \frac{1}{2} \right)}$$

5. (10 points) **Let** a, b, c **be positive numbers. Find the volume of the region inside** $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$**. Explain thoroughly.**

Let $u = \frac{x}{a}$, $v = \frac{y}{b}$, and $w = \frac{z}{c}$. The volume is equal to the integral over $u^2 + v^2 + w^2 \leq 1$ of the Jacobian and the Jacobian is abc. Of course, this is abc times the volume of a sphere of radius 1. The answer is $\boxed{\frac{4}{3}\pi abc}$.