Since the eighteenth century, solutions to the wave equation have been well studied (one learns these in most courses on differential equations). To indicate the wavelike nature of the solutions, for example, observe that for any function f,

$$\phi(t, x, y, z) = f(x - t)$$

solves the wave equation $\nabla^2\phi - (\partial^2\phi/\partial t^2) = 0$. This solution just propagates the graph of f like a wave; thus, we might conjecture that solutions of Maxwell's equations are wavelike in nature. Historically, all of this was Maxwell's great achievement, and it soon led to Hertz's discovery of radio waves.

Mathematics again shows its uncanny ability not only to *describe* but to *predict* natural phenomena.

exercises

In Exercises 1 to 4, verify the divergence theorem for the given region W, boundary ∂W oriented outward, and vector field $\mathbf{F}$.

1. $W = [0, 1] \times [0, 1] \times [0, 1]$
$\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

2. W as in Exercise 1, and $\mathbf{F} = zy\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

3. $W = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$ (the unit ball),
$\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

4. W as in Exercise 3, and $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$

5. Use the divergence theorem to calculate the flux of $\mathbf{F} = (x - y)\mathbf{i} + (y - z)\mathbf{j} + (z - x)\mathbf{k}$ out of the unit sphere.

6. Let $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$. Evaluate the surface integral of $\mathbf{F}$ over the unit sphere.

7. Evaluate $\iint_{\partial W} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and W is the unit cube (in the first octant). Perform the calculation directly and check by using the divergence theorem.

8. Repeat Exercise 7 for

(a) $\mathbf{F} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

(b) $\mathbf{F} = x^2\mathbf{i} + x^2\mathbf{j} + z^2\mathbf{k}$

9. Let $\mathbf{F} = y\mathbf{i} + z\mathbf{j} + xz\mathbf{k}$. Evaluate $\iint_{\partial W} \mathbf{F} \cdot d\mathbf{S}$ for each of the following regions W:

(a) $x^2 + y^2 \leq z \leq 1$

(b) $x^2 + y^2 \leq z \leq 1$ and $x \geq 0$

(c) $x^2 + y^2 \leq z \leq 1$ and $x \leq 0$

10. Repeat Exercise 9 for $\mathbf{F} = (x - y)\mathbf{i} + (y - z)\mathbf{j} + (z - x)\mathbf{k}$. [The solution to part (b) only is in the Study Guide to this text.]

11. Find the flux of the vector field $\mathbf{F} = (x - y^2)\mathbf{i} + y\mathbf{j} + x^3\mathbf{k}$ out of the rectangular solid $[0, 1] \times [1, 2] \times [1, 4]$.

12. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = 3xy^2\mathbf{i} + 3x^2y\mathbf{j} + z^3\mathbf{k}$ and S is the surface of the unit sphere.

13. Let W be the pyramid with top vertex $(0, 0, 1)$, and base vertices at $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(1, 1, 0)$. Let S be the two-dimensional closed surface bounding W, oriented outward from W. Use Gauss' theorem to calculate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where:

$$\mathbf{F}(x, y, z) = (x^2y, 3y^2z, 9z^2x).$$

14. Let W be the three-dimensional solid enclosed by the surfaces $x = y^2$, $x = 9$, $z = 0$, and $x = z$. Let S be the boundary of W. Use Gauss' theorem to find the flux of $\mathbf{F}(x, y, z) = (3x - 5y)i + (4z - 2y)j + (8yz)k$ across S: $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

15. Evaluate $\iint_{\partial W} \mathbf{F} \cdot \mathbf{n}\, dA$, where $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} - z\mathbf{k}$ and W is the unit cube in the first octant. Perform the calculation directly and check by using the divergence theorem.

16. Evaluate the surface integral $\iint_{\partial S} \mathbf{F} \cdot \mathbf{n}\, dA$, where $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{j} + z(x^2 + y^2)^2\mathbf{k}$ and ∂S is the surface of the cylinder $x^2 + y^2 \leq 1$, $0 \leq z \leq 1$.

17. Prove that

$$\iiint_W (\nabla f) \cdot \mathbf{F}\, dx\, dy\, dz = \iint_{\partial W} f\mathbf{F} \cdot \mathbf{n}\, dS - \iiint_W f\nabla \cdot \mathbf{F}\, dx\, dy\, dz.$$

18. Prove the identity

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G}).$$

19. Show that $\iiint_W (1/r^2)\, dx\, dy\, dz = \iint_{\partial W} (\mathbf{r} \cdot \mathbf{n}/r^2)\, dS$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

20. Fix vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^3$ and numbers ("charges") $q_1, \ldots, q_k$. Define the function ϕ by $\phi(x, y, z) = \sum_{i=1}^{k} q_i/(4\pi \|\mathbf{r} - \mathbf{v}_i\|)$, where $\mathbf{r} = (x, y, z)$. Show that for a closed surface S and $\mathbf{E} = -\nabla\phi$,

$$\iint_S \mathbf{E} \cdot d\mathbf{S} = Q,$$

where Q is the total charge inside S. (Assume that Gauss' law from Theorem 10 applies and that none of the charges are on S.)

21. Prove ***Green's identities***

$$\iint_{\partial W} f \nabla g \cdot \mathbf{n}\, dS = \iiint_W (f \nabla^2 g + \nabla f \cdot \nabla g)\, dV$$

and

$$\iint_{\partial W} (f \nabla g - g \nabla f) \cdot \mathbf{n}\, dS = \iiint_W (f \nabla^2 g - g \nabla^2 f)\, dV.$$

22. Suppose $\mathbf{F}$ satisfies div $\mathbf{F} = 0$ and curl $\mathbf{F} = \mathbf{0}$ on all of $\mathbb{R}^3$. Show that we can write $\mathbf{F} = \nabla f$, where $\nabla^2 f = 0$.

23. Let ρ be a continuous function on $\mathbb{R}^3$ such that $\rho(\mathbf{q}) = 0$ except for $\mathbf{q}$ in some region W. Let $\mathbf{q} \in W$ be denoted by $\mathbf{q} = (x, y, z)$. The ***potential*** of ρ is defined to be the function

$$\phi(\mathbf{p}) = \iiint_W \frac{\rho(\mathbf{q})}{4\pi \|\mathbf{p} - \mathbf{q}\|}\, dV(\mathbf{q}),$$

where $\|\mathbf{p} - \mathbf{q}\|$ is the distance between $\mathbf{p}$ and $\mathbf{q}$.

(a) Using the method of Theorem 10, show that

$$\iint_{\partial W} \nabla\phi \cdot \mathbf{n}\, dS = -\iiint_W \rho\, dV$$

for those regions W that can be partitioned into a finite union of symmetric elementary regions.

(b) Show that ϕ satisfies ***Poisson's equation***

$$\nabla^2 \phi = -\rho.$$

[HINT: Use part (a).] (Notice that if ρ is a charge density, then the integral defining ϕ may be thought of as the sum of the potential at $\mathbf{p}$ caused by point charges distributed over W according to the density ρ.)

24. Suppose $\mathbf{F}$ is tangent to the closed surface $S = \partial W$ of a region W. Prove that

$$\iiint_W (\text{div}\, \mathbf{F})\, dV = 0.$$

25. Use Gauss' law and symmetry to prove that the electric field due to a charge Q evenly spread over the surface of a sphere is the same outside the surface as the field from a point charge Q located at the center of the sphere. What is the field inside the sphere?

26. Reformulate Exercise 25 in terms of gravitational fields.

27. Show how Gauss' law can be used to solve part (b) of Exercise 29 in Section 8.3.

28. Let S be a closed surface. Use Gauss' theorem to show that if $\mathbf{F}$ is a C^2 vector field, then we have $\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = 0$.

29. Let S be the surface of region W. Show that

$$\iint_S \mathbf{r} \cdot \mathbf{n}\, dS = 3 \text{ volume } (W).$$

Explain this geometrically.

30. For a steady-state charge distribution and divergence-free current distribution, the electric and magnetic fields $\mathbf{E}(x, y, z)$ and $\mathbf{H}(x, y, z)$ satisfy

$$\nabla \times \mathbf{E} = \mathbf{0}, \quad \nabla \cdot \mathbf{H} = 0, \quad \nabla \cdot \mathbf{J} = 0, \quad \nabla \cdot \mathbf{E} = \rho, \quad \text{and} \quad \nabla \times \mathbf{H} = \mathbf{J}.$$

Here $\rho = \rho(x, y, z)$ and $\mathbf{J}(x, y, z)$ are assumed to be known. The radiation that the fields produce through a surface S is determined by a radiation flux density vector field, called the ***Poynting*** vector field,

$$\mathbf{P} = \mathbf{E} \times \mathbf{H}.$$

(a) If S is a *closed* surface, show that the radiation flux—that is, the flux of $\mathbf{P}$ through S—is given by

$$\iint_S \mathbf{P} \cdot d\mathbf{S} = -\iiint_V \mathbf{E} \cdot \mathbf{J}\, dV,$$

where V is the region enclosed by S.

(b) Examples of such fields are

$$\mathbf{E}(x, y, z) = z\mathbf{j} + y\mathbf{k} \quad \text{and} \quad \mathbf{H}(x, y, z) = -xy\mathbf{i} + x\mathbf{j} + yz\mathbf{k}.$$

In this case, find the flux of the Poynting vector through the hemispherical shell shown in Figure 8.4.9. (Notice that it is an *open* surface.)

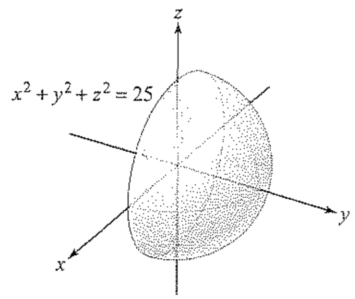

figure **8.4.9** The surface for Exercise 30.

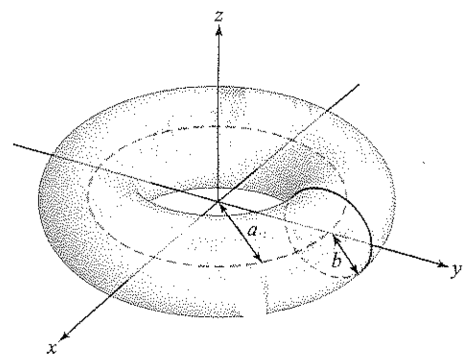

figure **8.4.10** The surface for Exercise 30(c).

(c) The fields of part (b) produce a Poynting vector field passing through the toroidal surface shown in Figure 8.4.10. What is the flux through this torus?

8.5 Differential Forms

The theory of differential forms provides an elegant way of formulating Green's, Stokes', and Gauss' theorems as one statement, the ***fundamental theorem of calculus***. The birth of the concept of a differential form is another dramatic example of how mathematics speaks to mathematicians and drives its own development. These three theorems are, in reality, generalizations of the fundamental theorem of calculus of Newton and Leibniz for functions of one variable,

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

to two and three dimensions.

Recall that Bernhard Riemann created the concept of n-dimensional spaces. If the fundamental theorem of calculus was truly *fundamental,* then it should generalize to arbitrary dimensions. But wait! The cross product, and therefore the curl, does not generalize to higher dimensions, as we remarked in footnote 3, in Section 1.3. Thus, some new idea is needed.

Recall that Hamilton searched for almost 15 years for his quaternions, which ultimately led to the discovery of the cross product. What is the nonexistence of a cross product in higher dimensions telling us? If the fundamental theorem of calculus is the core concept, this suggests the existence of a mathematical language in which it can be formulated in n-dimensions. In order to achieve this, mathematicians realized that they were forced to move away from vectors and on to the discovery of *dual* vectors and an entirely new mathematical object, a *differential form.* In this new language, all of the theorems of Green, Stokes, and Gauss have the same elegant and extraordinarily simple form.

Simply and very briefly stated, an expression of the type $P\,dx + Q\,dy$ is a 1-form, or a *differential one-form* on a region in the xy plane, and $F\,dx\;dy$ is a 2-form. Analogously, we can define the notion of an n-form. There is an operation d, which takes n-forms to $n+1$-forms. It is like a generalized curl and has the property that for $\omega = P\,dx + Q\,dy$,