c. Integral of a vector field **F**:

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}\cdot(\mathbf{T}_u \times \mathbf{T}_v)\,du\,dv$$

d. Vector surface element:

$$d\mathbf{S} = (\mathbf{T}_u \times \mathbf{T}_v)\,du\,dv = \mathbf{n}\,dS$$

2. **Graph**: $z = g(x, y)$

a. Integral of a scalar function f:

$$\iint_S f\,dS = \iint_D \frac{f(x, y, g(x, y))}{\cos\theta}\,dx\,dy$$

b. Scalar surface element:

$$dS = \frac{dx\,dy}{\cos\theta} = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\,dx\,dy,$$

where $\cos\theta = \mathbf{n}\cdot\mathbf{k}$, and $\mathbf{n}$ is a unit normal vector to the surface.

c. Integral of a vector field **F**:

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \left(-F_1\frac{\partial g}{\partial x} - F_2\frac{\partial g}{\partial y} + F_3\right)dx\,dy$$

d. Vector surface element:

$$d\mathbf{S} = \mathbf{n}\cdot dS = \left(-\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}\right)dx\,dy$$

3. **Sphere:** $x^2 + y^2 + z^2 = R^2$

a. Scalar surface element:

$$dS = R^2 \sin\phi\,d\phi\,d\theta$$

b. Vector surface element:

$$d\mathbf{S} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})R\sin\phi\,d\phi\,d\theta = \mathbf{r}R\sin\phi\,d\phi\,d\theta = \mathbf{n}R^2\sin\phi\,d\phi\,d\theta$$

exercises

1. Consider the closed surface S consisting of the graph $z = 1 - x^2 - y^2$ with $z \geq 0$, and also the unit disc in the xy plane. Give this surface an outer normal. Compute:

$$\iint_S \mathbf{F}\cdot d\mathbf{S}$$

where $\mathbf{F}(x, y, z) = (2x, 2y, z)$.

2. Evaluate the surface integral

$$\iint_S \mathbf{F}\cdot d\mathbf{S}$$

where $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z^2\mathbf{k}$ and S is the surface parameterized by $\Phi(u, v) = (2\sin u, 3\cos u, v)$, with $0 \leq u \leq 2\pi$ and $0 \leq v \leq 1$.

3. Let $\mathbf{F}(x, y, z) = (x, y, z)$. Evaluate

$$\iint_S \mathbf{F} \cdot d\mathbf{S},$$

where S is:

(a) the upper hemisphere of radius 3, centered at the origin.

(b) the entire sphere of radius 3, centered at the origin.

4. Let $\mathbf{F}(x, y, z) = 2x\mathbf{i} - 2y\mathbf{j} + z^2\mathbf{k}$. Evaluate

$$\iint_S \mathbf{F} \cdot d\mathbf{S},$$

where S is the cylinder $x^2 + y^2 = 4$ with $z \in [0, 1]$.

5. Let the temperature of a point in $\mathbb{R}^3$ be given by $T(x, y, z) = 3x^2 + 3z^2$. Compute the heat flux across the surface $x^2 + z^2 = 2$, $0 \le y \le 2$, if $k = 1$.

6. Compute the heat flux across the unit sphere S if $T(x, y, z) = x$. Can you interpret your answer physically?

7. Let S be the closed surface that consists of the hemisphere $x^2 + y^2 + z^2 = 1$, $z \ge 0$, and its base $x^2 + y^2 \le 1$, $z = 0$. Let $\mathbf{E}$ be the electric field defined by $\mathbf{E}(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$. Find the electric flux across S. (HINT: Break S into two pieces S_1 and S_2 and evaluate $\iint_{S_1} \mathbf{E} \cdot d\mathbf{S}$ and $\iint_{S_2} \mathbf{E} \cdot d\mathbf{S}$ separately.)

8. Let the velocity field of a fluid be described by $\mathbf{F} = \sqrt{y}\mathbf{i}$ (measured in meters per second). Compute how many cubic meters of fluid per second are crossing the surface $x^2 + z^2 = 1$, $0 \le y \le 1$, $0 \le x \le 1$.

9. Evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$, where S is the surface $x^2 + y^2 + 3z^2 = 1$, $z \le 0$ and $\mathbf{F}$ is the vector field $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + zx^3y^2\mathbf{k}$. (Let $\mathbf{n}$, the unit normal, be upward pointing.)

10. Evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$, where $\mathbf{F} = (x^2 + y - 4)\mathbf{i} + 3xy\mathbf{j} + (2xz + z^2)\mathbf{k}$ and S is the surface $x^2 + y^2 + z^2 = 16$, $z \ge 0$. (Let $\mathbf{n}$, the unit normal, be upward pointing.)

11. Calculate the integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is the entire surface of the solid half ball $x^2 + y^2 + z^2 \le 1$, $z \ge 0$, and $\mathbf{F} = (x + 3y^5)\mathbf{i} + (y + 10xz)\mathbf{j} + (z - xy)\mathbf{k}$. (Let S be oriented by the outward-pointing normal.)

12.* A restaurant is being built on the side of a mountain. The architect's plans are shown in Figure 7.6.11.

(a) The vertical curved wall of the restaurant is to be built of glass. What will be the surface area of this wall?

(b) To be large enough to be profitable, the consulting engineer informs the developer that the volume of the interior must exceed $\pi R^4/2$. For what R does the proposed structure satisfy this requirement?

(c) During a typical summer day, the environs of the restaurant are subject to a temperature field given by

$$T(x, y, z) = 3x^2 + (y - R)^2 + 16z^2.$$

A heat flux density $\mathbf{V} = -k\,\nabla T$ (k is a constant depending on the grade of insulation to be used) through all sides of the restaurant (including the top and the contact with the hill) produces a heat flux.

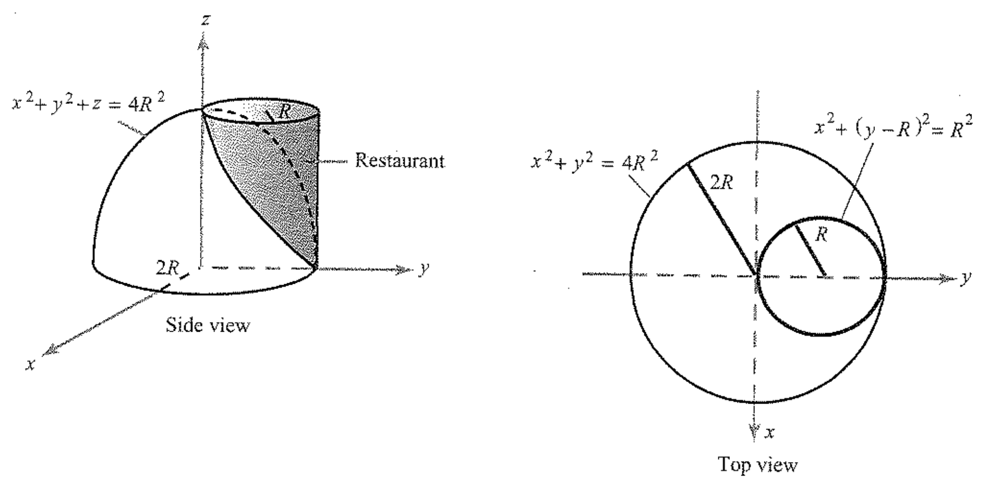

figure **7.6.11** Restaurant plans.

*The solution to this problem may be somewhat time-consuming.

What is this total heat flux? (Your answer will depend on R and k.)

13. Find the flux of the vector field $\mathbf{V}(x, y, z) = 3xy^2\mathbf{i} + 3x^2y\mathbf{j} + z^3\mathbf{k}$ out of the unit sphere.

14. Evaluate the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, dA$, where $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{j} + z(x^2 + y^2)^2\mathbf{k}$ and S is the surface of the cylinder $x^2 + y^2 \leq 1, 0 \leq z \leq 1$.

15. Let S be the surface of the unit sphere. Let $\mathbf{F}$ be a vector field and F_r its radial component. Prove that

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} F_r \sin\phi \, d\phi \, d\theta.$$

What is the corresponding formula for real-valued functions f?

16. Prove the following mean-value theorem for surface integrals: If $\mathbf{F}$ is a continuous vector field, then

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = [\mathbf{F}(\mathrm{Q}) \cdot \mathbf{n}(\mathrm{Q})]A(S)$$

for some point $\mathrm{Q} \in S$, where $A(S)$ is the area of S. [HINT: Prove it for real functions first, by reducing the problem to one of a double integral: Show that if $g \geq 0$, then

$$\iint_D fg\, dA = f(\mathrm{Q}) \iint_D g\, dA$$

for some $\mathrm{Q} \in D$ (do it by considering $(\iint_D fg\, dA)/(\iint_D g\, dA)$ and using the intermediate-value theorem).]

17. Work out a formula like that in Exercise 15 for integration over the surface of a cylinder.

18. Let S be a surface in $\mathbb{R}^3$ that is actually a subset D of the xy plane. Show that the integral of a scalar function $f(x, y, z)$ over S reduces to the double integral of $f(x, y, z)$ over D. What does the surface integral of a vector field over S become? (Make sure your answer is compatible with Example 6.)

19. Let the velocity field of a fluid be described by $\mathbf{F} = \mathbf{i} + x\mathbf{j} + z\mathbf{k}$ (measured in meters per second). Compute how many cubic meters of fluid per second are crossing the surface described by $x^2 + y^2 + z^2 = 1, z \geq 0$.

20. (a) A uniform fluid that flows vertically downward (heavy rain) is described by the vector field $\mathbf{F}(x, y, z) = (0, 0, -1)$. Find the total flux through the cone $z = (x^2 + y^2)^{1/2}, x^2 + y^2 \leq 1$.

(b) The rain is driven sideways by a strong wind so that it falls at a 45° angle, and it is described by $\mathbf{F}(x, y, z) = -(\sqrt{2}/2, 0, \sqrt{2}/2)$. Now what is the flux through the cone?

21. For $a > 0, b > 0, c > 0$, let S be the upper half ellipsoid

$$S = \left\{ (x, y, z) \,\middle|\, \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, z \geq 0 \right\},$$

with orientation determined by the upward normal. Compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F}(x, y, z) = (x^3, 0, 0)$.

22. If S is the upper hemisphere $\{(x, y, z) \mid x^2 + y^2 + z^2 = 1, z \geq 0\}$ oriented by the normal pointing out of the sphere, compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$ for parts (a) and (b).

(a) $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j}$

(b) $\mathbf{F}(x, y, z) = y\mathbf{i} + x\mathbf{j}$

(c) For each of these vector fields, compute $\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$ and $\int_C \mathbf{F} \cdot d\mathbf{s}$, where C is the unit circle in the xy plane traversed in the counterclockwise direction (as viewed from the positive z axis). (Notice that C is the boundary of S. The phenomenon illustrated here will be studied more thoroughly in the next chapter, using Stokes' theorem.)

7.7 Applications to Differential Geometry, Physics, and Forms of Life*

In the first half of the nineteenth century, the great German mathematician Karl Friedrich Gauss developed a theory of curved surfaces in $\mathbb{R}^3$. More than a century earlier, Isaac Newton had defined a measure of the curvature of a space curve, and Gauss was able to find extensions of this idea of curvature that would apply to surfaces. In so doing, Gauss made several remarkable discoveries.

*This section can be skipped on a first reading without loss of continuity.