exercises

1. Evaluate the integral of the function $f(x, y, z) = x + y$ over the surface S given by:

$$\Phi(u, v) = (2u \cos v, 2u \sin v, u), \quad u \in [0, 4], v \in [0, \pi]$$

2. Evaluate the integral of the function $f(x, y, z) = z + 6$ over the surface S given by:

$$\Phi(u, v) = \left(u, \frac{v}{3}, v\right), \quad u \in [0, 2], v \in [0, 3].$$

3. Evaluate the integral

$$\iint_S (3x - 2y + z)\, dS,$$

where S is the portion of the plane $2x + 3y + z = 6$ that lies in the first octant.

4. Evaluate the integral

$$\iint_S (x + z)\, dS,$$

where S is the part of the cylinder $y^2 + z^2 = 4$ with $x \in [0, 5]$.

5. Let S be the surface defined by

$$\Phi(u, v) = (u + v, u - v, uv).$$

   (a) Show that the image of S is in the graph of the surface $4z = x^2 - y^2$.

   (b) Evaluate $\iint_S x\, dS$ for all points on the graph S, over $x^2 + y^2 \leq 1$.

6. Evaluate the integral

$$\iint_S (x^2 z + y^2 z)\, dS,$$

where S is the part of the plane $z = 4 + x + y$ that lies inside the cylinder $x^2 + y^2 = 4$.

7. Compute $\iint_S xy\, dS$, where S is the surface of the tetrahedron with sides $z = 0$, $y = 0$, $x + z = 1$, and $x = y$.

8. Evaluate $\iint_S xyz\, dS$, where S is the triangle with vertices $(1, 0, 0)$, $(0, 2, 0)$, and $(0, 1, 1)$.

9. Evaluate $\iint_S z\, dS$, where S is the upper hemisphere of radius a, that is, the set of (x, y, z) with $z = \sqrt{a^2 - x^2 - y^2}$.

10. Evaluate $\iint_S (x + y + z)\, dS$, where S is the boundary of the unit ball B; that is, S is the set of (x, y, z) with $x^2 + y^2 + z^2 = 1$. (HINT: Use the symmetry of the problem.)

11. (a) Compute the area of the portion of the cone $x^2 + y^2 = z^2$ with $z \geq 0$ that is inside the sphere $x^2 + y^2 + z^2 = 2Rz$, where R is a positive constant.

    (b) What is the area of that portion of the sphere that is inside the cone?

12. Verify that in spherical coordinates, on a sphere of radius R,

$$\|\mathbf{T}_\phi \times \mathbf{T}_\theta\|\, d\phi\, d\theta = R^2 \sin\phi\, d\phi\, d\theta.$$

13. Evaluate $\iint_S z\, dS$, where S is the surface $z = x^2 + y^2$, $x^2 + y^2 \leq 1$.

14. Evaluate the surface integral $\iint_S z^2\, dS$, where S is the boundary of the cube $C = [-1, 1] \times [-1, 1] \times [-1, 1]$. (HINT: Do each face separately and add the results.)

15. Find the mass of a spherical surface S of radius R such that at each point $(x, y, z) \in S$ the mass density is equal to the distance of (x, y, z) to some fixed point $(x_0, y_0, z_0) \in S$.

16. A metallic surface S is in the shape of a hemisphere $z = \sqrt{R^2 - x^2 - y^2}$, where (x, y) satisfies $0 \leq x^2 + y^2 \leq R^2$. The mass density at $(x, y, z) \in S$ is given by $m(x, y, z) = x^2 + y^2$. Find the total mass of S.

17. Let S be the sphere of radius R.

    (a) Argue by symmetry that

$$\iint_S x^2\, dS = \iint_S y^2\, dS = \iint_S z^2\, dS.$$

    (b) Use this fact and some clever thinking to evaluate, with very little computation, the integral

$$\iint_S x^2\, dS.$$

    (c) Does this help in Exercise 16?

18. (a) Use Riemann sums to justify the formula

$$\frac{1}{A(S)} \iint_S f(x, y, z)\, dS$$

for the *average value* of f over the surface S.

(b) In Example 3 of this section, show that the average of $f(x, y, z) = z^2$ over the sphere is $1/3$.

(c) Define the ***center of gravity*** $(\bar{x}, \bar{y}, \bar{z})$ of a surface S to be such that $\bar{x}$, $\bar{y}$, and $\bar{z}$ are the average values of the x, y, and z coordinates on S. Show that the center of gravity of the triangle in Example 4 of this section is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

19. Find the average value of $f(x, y, z) = x + z^2$ on the truncated cone $z^2 = x^2 + y^2$, with $1 \le z \le 4$.

20. Evaluate the integral

$$\iint_S (1 - z)\, dS,$$

where S is the graph of $z = 1 - x^2 - y^2$, with $x^2 + y^2 \le 1$.

21. Find the x, y, and z coordinates of the center of gravity of the octant of the solid sphere of radius R and centered at the origin determined by $x \ge 0, y \ge 0, z \ge 0$. (HINT: Write this octant as a parametrized surface—see Example 3 of this section and Exercise 18.)

22. Find the z coordinate of the center of gravity (the average z coordinate) of the surface of a hemisphere ($z \le 0$) with radius r (see Exercise 18). Argue by symmetry that the average x and y coordinates are both zero.

23. Let $\mathbf{\Phi}: D \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a parametrization of a surface S defined by

$$x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v).$$

(a) Let

$$\frac{\partial \mathbf{\Phi}}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) \quad \text{and} \quad \frac{\partial \mathbf{\Phi}}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right),$$

that is, $\partial\mathbf{\Phi}/\partial u = \mathbf{T}_u$ and $\partial\mathbf{\Phi}/\partial v = \mathbf{T}_v$, and set

$$E = \left\|\frac{\partial \mathbf{\Phi}}{\partial u}\right\|^2, \qquad F = \frac{\partial \mathbf{\Phi}}{\partial u} \cdot \frac{\partial \mathbf{\Phi}}{\partial v}, \qquad G = \left\|\frac{\partial \mathbf{\Phi}}{\partial v}\right\|^2.$$

Show that

$$\sqrt{EG - F^2} = \|\mathbf{T}_u \times \mathbf{T}_v\|,$$

and that the surface area of S is

$$A(S) = \iint_D \sqrt{EG - F^2}\, du\, dv.$$

In this notation, how can we express $\iint_S f\, dS$ for a general function of f?

(b) What does the formula for $A(S)$ become if the vectors $\partial\mathbf{\Phi}/\partial u$ and $\partial\mathbf{\Phi}/\partial v$ are orthogonal?

(c) Use parts (a) and (b) to compute the surface area of a sphere of radius a.

24. ***Dirichlet's functional*** for a parametrized surface $\mathbf{\Phi}: D \to \mathbb{R}^3$ is defined by[11]

$$J(\mathbf{\Phi}) = \frac{1}{2}\iint_D \left(\left\|\frac{\partial \mathbf{\Phi}}{\partial u}\right\|^2 + \left\|\frac{\partial \mathbf{\Phi}}{\partial v}\right\|^2\right) du\, dv.$$

Use Exercise 23 to argue that the area $A(\mathbf{\Phi}) \le J(\mathbf{\Phi})$ and equality holds if

$$\text{(a)} \quad \left\|\frac{\partial \mathbf{\Phi}}{\partial u}\right\|^2 = \left\|\frac{\partial \mathbf{\Phi}}{\partial v}\right\|^2 \qquad \text{and} \qquad \text{(b)} \quad \frac{\partial \mathbf{\Phi}}{\partial u} \cdot \frac{\partial \mathbf{\Phi}}{\partial v} = 0.$$

Compare these equations with Exercise 23 and the remarks at the end of Section 7.4. A parametrization $\mathbf{\Phi}$ that satisfies conditions (a) and (b) is said to be ***conformal***.

25. Let $D \subset \mathbb{R}^2$ and $\mathbf{\Phi}: D \to \mathbb{R}^2$ be a smooth function $\mathbf{\Phi}(u, v) = (x(u, v), y(u, v))$ satisfying conditions (a) and (b) of Exercise 16 and assume that

$$\det \begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix} > 0.$$

Show that x and y satisfy the ***Cauchy–Riemann equations*** $\partial x/\partial u = \partial y/\partial v$, $\partial x/\partial v = -\partial y/\partial u$. Conclude that $\nabla^2\mathbf{\Phi} = 0$ (i.e., each component of $\mathbf{\Phi}$ is harmonic).

26. Let S be a sphere of radius r and $\mathbf{p}$ be a point inside or outside the sphere (but not on it). Show that

$$\iint_S \frac{1}{\|\mathbf{x} - \mathbf{p}\|}\, dS = \begin{cases} 4\pi r & \text{if } \mathbf{p} \text{ is inside } S \\ 4\pi r^2/d & \text{if } \mathbf{p} \text{ is outside } S, \end{cases}$$

[11] Dirichlet's functional played a major role in the mathematics of the nineteenth century. The mathematician Georg Friedrich Bernhard Riemann (1826–1866) used it to develop his complex function theory and to give a proof of the famous Riemann mapping theorem. Today it is still used extensively as a tool in the study of partial differential equations.

where d is the distance from $\mathbf{p}$ to the center of the sphere and the integration is over the sphere. [Hint: Assume $\mathbf{p}$ is on the z-axis. Then change variables and evaluate. Why is this assumption on $\mathbf{p}$ justified?]

27. Find the surface area of that part of the cylinder $x^2 + z^2 = a^2$ that is inside the cylinder $x^2 + y^2 = 2ay$ and also in the positive octant ($x \geq 0, y \geq 0, z \geq 0$). Assume $a > 0$.

28. Let a surface S be defined implicitly by $F(x, y, z) = 0$ for (x, y) in a domain D of $\mathbb{R}^2$. Show that

$$\iint_S \left|\frac{\partial F}{\partial z}\right| dS = \iint_D \sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}\, dx\, dy.$$

Compare with Exercise 22 of Section 7.4.

7.6 Surface Integrals of Vector Fields

The goal of this section is to develop the notion of the integral of a vector field over a surface. Recall that the definition of the line integral of a vector field was motivated by the fundamental physical notion of *work*. Similarly, there is a basic physical notion of *flux* that motivates the definition of the surface integral of a vector field.

For example, if the vector field is the velocity field of a fluid (perhaps the velocity field of a flowing river), and we put an imagined mathematical surface into the fluid, we can ask: "What is the rate at which fluid is crossing the given surface (measured in, say, cubic meters per second)?" The answer is given by the surface integral of the fluid velocity vector field over the surface.

We shall come back to the physical interpretation shortly and reconcile it with the formal definition that we give first.

Definition of the Surface Integral

We now define the integral of a vector field, denoted $\mathbf{F}$, over a surface S. We first give the definition and later in this section give its physical interpretation. This can also be used as a *motivation* for the definition if you so desire. Also, we shall start with a parametrized surface $\mathbf{\Phi}$ and later study the question of independence of parametrization.

Definition The Surface Integral of Vector Fields Let $\mathbf{F}$ be a vector field defined on S, the image of a parametrized surface $\mathbf{\Phi}$. The ***surface integral*** of $\mathbf{F}$ over $\mathbf{\Phi}$, denoted by

$$\iint_{\mathbf{\Phi}} \mathbf{F} \cdot d\mathbf{S},$$

is defined by (see Figure 7.6.1)

$$\iint_{\mathbf{\Phi}} \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{T}_u \times \mathbf{T}_v)\, du\, dv.$$