Answers to these questions involve central concepts in our book, namely, problems of maxima and minima, and in the case of soap bubbles problems of area and volume. Soap bubbles are round because nature is economical. The spherical shape is the shape of the unique surface of least possible area containing a fixed volume (which in the case of a bubble is air).

Mathematical problems of this kind belong to a subject called The Calculus of Variations, a subject almost as old as The Calculus itself.

For more information, consult our Internet supplement or the book *The Parsimonious Universe* by Hildebrandt and Tromba (Springer-Verlag, 1995).

exercises

1. Find the surface area of the unit sphere S represented parametrically by $\mathbf{\Phi}: D \to S \subset \mathbb{R}^3$, where D is the rectangle $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$ and $\mathbf{\Phi}$ is given by the equations

$$x = \cos\theta \sin\phi, \qquad y = \sin\theta \sin\phi, \qquad z = \cos\phi.$$

Note that we can represent the entire sphere parametrically, but we cannot represent it in the form $z = f(x, y)$.

2. In Exercise 1, what happens if we allow ϕ to vary from $-\pi/2$ to $\pi/2$? From 0 to 2π? Why do we obtain different answers?

3. Find the area of the helicoid in Example 2 if the domain D is $0 \le r \le 1$ and $0 \le \theta \le 3\pi$.

4. The torus T can be represented parametrically by the function $\mathbf{\Phi}: D \to \mathbb{R}^3$, where $\mathbf{\Phi}$ is given by the coordinate functions $x = (R + \cos\phi)\cos\theta$, $y = (R + \cos\phi)\sin\theta$, $z = \sin\phi$; D is the rectangle $[0, 2\pi] \times [0, 2\pi]$, that is, $0 \le \theta \le 2\pi$, $0 \le \phi \le 2\pi$; and $R > 1$ is fixed (see Figure 7.4.6). Show that $A(T) = (2\pi)^2 R$, first by using formula (3) and then by using formula (6).

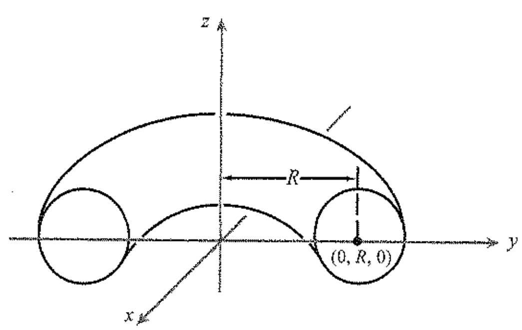

figure **7.4.6** A cross section of a torus.

5. Let $\Phi(u, v) = (e^u \cos v, e^u \sin v, v)$ be a mapping from $D = [0, 1] \times [0, \pi]$ in the uv plane onto a surface S in xyz space.

(a) Find $T_u \times T_v$.

(b) Find the equation for the tangent plane to S when $(u, v) = (0, \frac{\pi}{2})$.

(c) Find the area of $\Phi(D)$.

6. Find the area of the surface defined by $z = xy$ and $x^2 + y^2 \le 2$.

7. Use a surface integral to find the area of the triangle T in $\mathbb{R}^3$ with vertices at (1, 1, 0), (2, 1, 2), and (2, 3, 3). Verify your answer by finding the lengths of the sides and using classical geometry.
[HINT: Write the triangle as the graph $z = g(x, y)$ over a triangle T^* in the xy plane.]

8. Use a surface integral to find the area of the quadrilateral D in $\mathbb{R}^3$ with vertices at $(-1, 1, 2)$, (1, 1, 2), (0, 3, 5), and (5, 3, 5). Verify your answer by finding the lengths of the sides and using classical geometry. [HINT: See the hint in the previous problem.]

9. Let $\mathbf{\Phi}(u, v) = (u - v, u + v, uv)$ and let D be the unit disc in the uv plane. Find the area of $\mathbf{\Phi}(D)$.

10. Find the area of the portion of the unit sphere that is cut out by the cone $z \ge \sqrt{x^2 + y^2}$ (see Exercise 1).

11. Show that the surface $x = 1/\sqrt{y^2 + z^2}$, where $1 \le x < \infty$, can be filled but not painted!

12. Find a parametrization of the surface $x^2 - y^2 = 1$, where $x > 0$, $-1 \le y \le 1$ and $0 \le z \le 1$. Use your answer to express the area of the surface as an integral.

13. Represent the ellipsoid E:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

parametrically and write out the integral for its surface area $A(E)$. (Do not evaluate the integral.)

14. Let the curve $y = f(x)$, $a \le x \le b$, be rotated about the y axis. Show that the area of the surface swept out is given by equation (6); that is,

$$A = 2\pi \int_a^b |x|\sqrt{1 + [f'(x)]^2}\, dx.$$

Interpret the formula geometrically using arc length and slant height.

15. Find the area of the surface obtained by rotating the curve $y = x^2$, $0 \le x \le 1$, about the y axis.

16. Use formula (4) to compute the surface area of the cone in Example 1.

17. Find the area of the surface defined by $x + y + z = 1$, $x^2 + 2y^2 \le 1$.

18. Show that for the vectors $\mathbf{T}_u$ and $\mathbf{T}_v$, we have the formula

$$\|\mathbf{T}_u \times \mathbf{T}_v\| = \sqrt{\left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2}.$$

19. Compute the area of the surface given by

$$x = r\cos\theta, \quad y = 2r\cos\theta, \quad z = \theta,$$
$$0 \le r \le 1, \quad 0 \le \theta \le 2\pi.$$

Sketch.

20. Prove *Pappus' theorem*: Let $\mathbf{c}\colon [a, b] \to \mathbb{R}^2$ be a C^1 path whose image lies in the right half plane and is a simple closed curve. The area of the lateral surface generated by rotating the image of $\mathbf{c}$ about the y axis is equal to $2\pi\bar{x}l(\mathbf{c})$, where $\bar{x}$ is the average value of the x coordinates of points on $\mathbf{c}$ and $l(\mathbf{c})$ is the length of $\mathbf{c}$. (See Exercises 16 to 19, Section 7.1, for a discussion of average values.)

21. The cylinder $x^2 + y^2 = x$ divides the unit sphere S into two regions S_1 and S_2, where S_1 is inside the cylinder and S_2 outside. Find the ratio of areas $A(S_2)/A(S_1)$.

22. Suppose a surface S that is the graph of a function $z = f(x, y)$, where $(x, y) \in D \subset \mathbb{R}^2$ can also be described as the set of $(x, y, z) \in \mathbb{R}^3$ with $F(x, y, z) = 0$ (a level surface). Derive a formula for $A(S)$ that involves only F.

23. Calculate the area of the frustum shown in Figure 7.4.7 using (a) geometry alone and, second, (b) a surface-area formula.

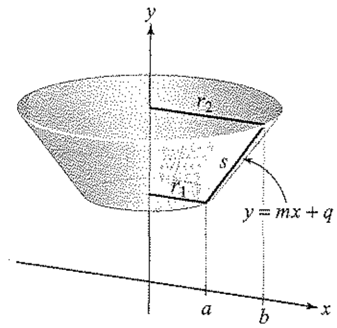

figure **7.4.7** A line segment revolved around the y axis becomes a frustum of a cone.

24. A cylindrical hole of radius 1 is bored through a solid ball of radius 2 to form a ring coupler, as shown in Figure 7.4.8. Find the volume and outer surface area of this coupler.

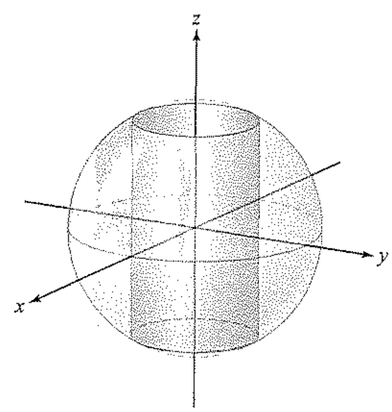

figure **7.4.8** Find the outer surface area and volume of the shaded region.

25. Find the area of the graph of the function $f(x, y) = \frac{2}{3}(x^{3/2} + y^{3/2})$ that lies over the domain $[0, 1] \times [0, 1]$.

26. Express the surface area of the following graphs over the indicated region D as a double integral. Do not evaluate.

(a) $(x + 2y)^2$; $D = [-1, 2] \times [0, 2]$

(b) $xy + x/(y + 1)$; $D = [1, 4] \times [1, 2]$

(c) $xy^3 e^{x^2y^2}$; $D =$ unit circle centered at the origin

(d) $y^3 \cos^2 x$; $D =$ triangle with vertices $(-1, 1)$, $(0, 2)$, and $(1, 1)$

27. Show that the surface area of the upper hemisphere of radius R, $z = \sqrt{R^2 - x^2 - y^2}$, can be computed by formula (4), evaluated as an improper integral.