exercises

1. Evaluate each of the following integrals if $R = [0, 1] \times [0, 1]$.

(a) $\displaystyle\iint_R (x^3 + y^2)\, dA$

(b) $\displaystyle\iint_R ye^{xy}\, dA$

(c) $\displaystyle\iint_R (xy)^2 \cos x^3\, dA$

(d) $\displaystyle\iint_R \ln[(x+1)(y+1)]\, dA$

2. Evaluate each of the following integrals if $R = [0, 1] \times [0, 1]$.

(a) $\displaystyle\iint_R (x^m y^n)\, dx\, dy$, where $m, n > 0$

(b) $\displaystyle\iint_R (ax + by + c)\, dx\, dy$

(c) $\displaystyle\iint_R \sin(x + y)\, dx\, dy$

(d) $\displaystyle\iint_R (x^2 + 2xy + y\sqrt{x})\, dx\, dy$

3. Evaluate over the region R:

$$\iint_R \frac{yx^3}{y^2+2}\, dy\, dx, \qquad R\colon [0, 2] \times [-1, 1].$$

4. Evaluate over the region R:

$$\iint_R \frac{y}{1+x^2}\, dx\, dy, \qquad R\colon [0, 1] \times [-2, 2].$$

5. Sketch the solid whose volume is given by:

$$\int_0^1 \int_0^1 (5 - x - y)\, dy\, dx.$$

6. Sketch the solid whose volume is given by:

$$\int_0^3 \int_0^2 (9 + x^2 + y^2)\, dx\, dy.$$

7. Compute the volume of the region over the rectangle $[0, 1] \times [0, 1]$ and under the graph of $z = xy$.

8. Compute the volume of the solid bounded by the xz plane, the yz plane, the xy plane, the planes $x = 1$ and $y = 1$, and the surface $z = x^2 + y^4$.

9. Let f be continuous on $[a, b]$ and g continuous on $[c, d]$. Show that

$$\iint_R [f(x)g(y)]\, dx\, dy = \left[\int_a^b f(x)\, dx\right]\left[\int_c^d g(y)\, dy\right],$$

where $R = [a, b] \times [c, d]$.

10. Compute the volume of the solid bounded by the surface $z = \sin y$, the planes $x = 1$, $x = 0$, $y = 0$, and $y = \pi/2$, and the xy plane.

11. Compute the volume of the solid bounded by the graph $z = x^2 + y$, the rectangle $R = [0, 1] \times [1, 2]$, and the "vertical sides" of R.

12. Let f be continuous on $R = [a, b] \times [c, d]$; for $a < x < b$, $c < y < d$, define

$$F(x, y) = \int_a^x \int_c^y f(u, v)\, dv\, du.$$

Show that $\partial^2 F/\partial x\, \partial y = \partial^2 F/\partial y\, \partial x = f(x, y)$. Use this example to discuss the relationship between Fubini's theorem and the equality of mixed partial derivatives.

13. Consider the integral in 2(a) as a function of m and n; that is,

$$f(m, n) := \iint_R x^m y^n\, dx\, dy.$$

Evaluate $\lim_{m,n\to\infty} f(m, n)$.

14. Let:

$$f(m, n) := \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \cos nx \sin my\, dx\, dy.$$

Show that $\lim_{m,n\to\infty} f(m, n) = 0$.

15. Let $f\colon [0, 1] \times [0, 1] \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} 1 & x \text{ rational} \\ 2y & x \text{ irrational.} \end{cases}$$

Show that the iterated integral $\int_0^1 \left[\int_0^1 f(x, y)\, dy\right] dx$ exists but that f is not integrable.

16. Express $\iint_R \cosh xy\, dx\, dy$ as a convergent sequence, where $R = [0, 1] \times [0, 1]$.

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17. Although Fubini's theorem holds for most functions met in practice, we must still exercise some caution. This exercise gives a function for which it fails. By using a substitution involving the tangent function, show that

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2}\, dy\, dx = \frac{\pi}{4},$$

yet

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2}\, dx\, dy = -\frac{\pi}{4}.$$

Why does this not contradict Theorem 3 or 3′?

18. Let f be continuous, $f \geq 0$, on the rectangle R. If $\iint_R f\, dA = 0$, prove that $f = 0$ on R.

5.3 The Double Integral Over More General Regions

Our goal in this section is twofold: First, we wish to define the double integral of a function $f(x, y)$ over regions D more general than rectangles; second, we want to develop a technique for evaluating this type of integral. To accomplish this, we shall define three special types of subsets of the xy plane, and then extend the notion of the double integral to them.

Elementary Regions

Suppose we are given two continuous real-valued functions $\phi_1\colon [a, b] \to \mathbb{R}$ and $\phi_2\colon [a, b] \to \mathbb{R}$ that satisfy $\phi_1(x) \leq \phi_2(x)$ for all $x \in [a, b]$. Let D be the set of all points (x, y) such that $x \in [a, b]$ and $\phi_1(x) \leq y \leq \phi_2(x)$. This region D is said to be ***y-simple***. Figure 5.3.1 shows various examples of y-simple regions. The curves and straight-line segments that bound the region together constitute the ***boundary*** of D, denoted ∂D. We use the phrase "y-simple" because the region is described in a relatively simple way, using y as a function of x.

We say that a region D is ***x-simple*** if there are continuous functions ψ_1 and ψ_2 defined on $[c, d]$ such that D is the set of points (x, y) satisfying

$$y \in [c, d] \qquad \text{and} \qquad \psi_1(y) \leq x \leq \psi_2(y),$$

where $\psi_1(y) \leq \psi_2(y)$ for all $y \in [c, d]$. Again, the curves that bound the region D constitute its boundary ∂D. Some examples of x-simple regions are shown in Figure 5.3.2. In this situation, x is the distinguished variable, given as a function of y. Thus, the phrase x-simple is appropriate.

Finally, a ***simple*** region is one that is both x- and y-simple; that is, a simple region can be described as both an x-simple region and a y-simple region. An example of a simple region is a unit disk (see Figure 5.3.3).

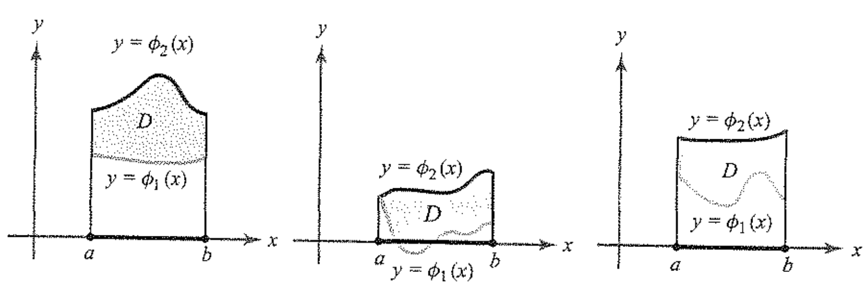

figure **5.3.1** Some y-simple regions.