PRINT Your Name:____________________________________

Quiz for March 4, 2005

1. Let $F \subseteq E$ be fields with E a finite dimensional vector space over F. Let R be a ring with $F \subseteq R \subseteq E$. Prove that R is a field.

ANSWER: Take $u \in R$, with $u \neq 0$. I will prove that $F[u]$ is a field. This is enough because, once I know that $F[u]$ is a field, then I will know that the inverse of u is in $F[u] \subseteq R$. Consider the ring homomorphism $\varphi\colon F[x] \to F[u]$, which is given by $\varphi(f(x)) = f(u)$. It is clear that φ is onto. The kernel of φ is not zero because $F[u]$ is a finite dimensional vector space over the field F. So the kernel of φ is a non-zero ideal of the PID $F[x]$. Apply the First Isomorphism Theorem to see that $\frac{F[x]}{\ker\varphi} \cong R[u]$. The ring $R[u]$ is a domain (because it is a subring of a field); hence, $\ker\varphi$ is a prime ideal in $F[x]$. But every non-zero prime ideal in a PID is also a maximal ideal. We conclude that $\frac{F[x]}{\ker\varphi}$ is a field; and therefore, $R[u]$ is a field as we desired.