PRINT Your Name:____________________________________

Quiz for February 11, 2005

Definition. Let I be an ideal of the ring R, with $I \neq R$. The ideal I is a *prime ideal* of R if, whenever a and b are in R with $ab \in I$, then $a \in I$ or $b \in I$.

Definition. Let I be an ideal of the ring R, with $I \neq R$. The ideal I is a *maximal ideal* of R if R is the only ideal of R which properly contains I.

Definition. The domain R is a *Principal Ideal Domain* if every ideal in R is principal.

1. Prove that every non-zero prime ideal in a Principal Ideal Domain is a maximal ideal.

ANSWER: Let I be a non-zero prime ideal of the Principal Ideal Domain R. We know that $I = (r)$ for some element r of R. Let J be an ideal of R with $I \subseteq J \subseteq R$. The ring R is a PID, so $J = (s)$ for some element s of R. We have $r \in I \subseteq J = (s)$; so, $r = st$ for some element t in R. The product st is in the prime ideal I. It follows that either $s \in I$ or $t \in I$.
Case 1. If $s \in I$, then $s = ar$ for some element a in R and $r = st = art$. The ring R is a domain; hence, $1 = at$. In other words, a is a unit and $I = J$.
Case 2. If $t \in I$, then $t = rb$ for some $b \in R$ and $r = st = srb$. The ring R is a domain; hence, $1 = sb$. In this case s is a unit and $J = R$.

We have shown that there do not exist any ideals J of R with $I \subsetneq J \subsetneq R$; and therefore, I is a maximal ideal of R.