Homework Problems Math 547 January 29, 2005 CORRECTED.

Definition. Let I be an ideal of the ring R, with $I \neq R$. The ideal I is a *prime ideal* of R if, whenever a and b are in R with $ab \in I$, then $a \in I$ or $b \in I$.

Definition. Let I be an ideal of the ring R, with $I \neq R$. The ideal I is a *maximal ideal* of R if R is the only ideal of R which properly contains I.

Definition. The domain R is a *Principal Ideal Domain* if every ideal in R is principal.

1. (a) Prove that every maximal ideal is a prime ideal.
   (b) Give an example of a non-zero prime ideal which is not a maximal ideal.
   (c) Prove that every non-zero prime ideal in a Principal Ideal Domain is a maximal ideal.

Definition. The element u of the ring R is called a *unit* if u has a multiplicative inverse in R.

Definition. The element r of the ring R is called *irreducible* if r is not zero, r is not a unit, and whenever $r = st$ in R, then either s is a unit or t is a unit.

2. Let R be the ring $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \in \mathbb{C} \mid a, b \in \mathbb{Z}\}$.
   (a) Prove that 1 and -1 are the only units in R.
   (b) Prove that 2, 3, $1+\sqrt{-5}$, and $1-\sqrt{-5}$ all are irreducible elements of the ring R.
   (c) Notice that none of the elements of R from (b) is a unit of R times a different element from (b).
   (d) Show that 6 can be factored into irreducible elements of R in two different ways.

3. (a) (This is called Gauss' Lemma.) Let $f(x)$ and $g(x)$ be polynomials in $\mathbb{Z}[x]$. Suppose that the coefficients of $f(x)$ are relatively prime. Suppose that the coefficients of $g(x)$ are relatively prime. Prove that the coefficients of $f(x)g(x)$ are relatively prime.
   (b) Let $f(x)$ be a polynomial in $\mathbb{Z}[x]$ with relatively prime coefficients. Suppose $f(x)$ is irreducible in $\mathbb{Z}[x]$. Prove $f(x)$ is irreducible in $\mathbb{Q}[x]$.
   (c) (This is called the Eisenstein Criteria.) Let $f(x) = a_0 + a_1x + \dots a_nx^n$ be a polynomial in $\mathbb{Z}[x]$ with relatively prime coefficients. Suppose that p is a prime integer such that p divides $a_0, a_1, \dots, a_{n-1}$, p^2 does not divide a_0, and p does not divide a_n. Prove that $f(x)$ is an irreducible polynomial in $\mathbb{Q}[x]$.
   (d) Prove that $x^2 - 2$, $x^5 - 2$, $x^{15} + 3x + 6$ are irreducible polynomials in $\mathbb{Q}[x]$.
   (e) Let p be a prime integer. Prove that $1+x+x^2+\dots+x^{p-1}$ is an irreducible polynomial in $\mathbb{Q}[x]$.