Math 547, Exam 1, Spring, 2005

The exam is worth 50 points.

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Take enough space for each problem. Turn in your solutions in the order: problem 1, problem 2, ... ; although, by using enough paper, you can do the problems in any order that suits you.

I will e-mail your grade to you.

I will post the solutions on my website later today.

1. **(7 points) Let**

$$S=\{\tfrac{a}{b}\in\mathbb{Q}\mid \text{either } a=0\text{, or } a \text{ and } b \text{ are relatively prime integers and } 3 \text{ does not divide } b\}.$$

Is S a subring of $\mathbb{Q}$? Explain.

YES. The set S contains 0 and 1. If $\frac{a}{b}$ is in S, then $-\frac{a}{b}$ is also in S. The set S is closed under both addition and multiplication. If $\frac{a}{b}$ and $\frac{c}{d}$ are in S, then the demoninator of $\frac{a}{b}+\frac{c}{d}$ is a factor of bd. We had b and d were each relatively prime to 3. We conclude that every factor of bd is also relatively prime to 3. Of course the product $\frac{a}{b}\frac{c}{d}$ is in S for the same reason.

2. **(7 points) Let**

$$S=\{\tfrac{a}{b}\in\mathbb{Q}\mid \text{either } a=0\text{, or } a \text{ and } b \text{ are relatively prime integers and } 9 \text{ does not divide } b\}.$$

Is S a subring of $\mathbb{Q}$? Explain.

NO. The set S is not closed under multiplication since $\frac{1}{3}\in S$; but $\frac{1}{3}\frac{1}{3}=\frac{1}{9}\notin S$.

3. **(7 points) Let $\phi\colon \mathbb{Z}[x]\to\mathbb{Z}[x]$ be the function which is given by $\phi(f(x))=(f(x))^2$. Is ϕ a ring homomorphism? Explain.**

NO. We see that $\phi(1+1)=\phi(2)=2^2=4$. But, $\phi(1)+\phi(1)=1^2+1^2=1+1=2$.

4. **(7 points) Let $\phi\colon \frac{\mathbb{Z}}{(2)}[x]\to\frac{\mathbb{Z}}{(2)}[x]$ be the function which is given by $\phi(f(x))=(f(x))^2$. Is ϕ a ring homomorphism? Explain.**

YES. The ring $R=\frac{\mathbb{Z}}{(2)}[x]$ has characteristic two. It follows that

$$\phi(r_1+r_2)=(r_1+r_2)^2=r_1^2+2r_1r_2+r_2^2=r_1^2+r_2^2=\phi(r_1)+\phi(r_2)$$

for all r_1 and r_2 in R. It is also clear that $\phi(1)=1^2=1$ and

$$\phi(r_1r_2)=(r_1r_2)^2=r_1^2r_2^2=\phi(r_1)\phi(r_2).$$

5. **(7 points) Let $\phi\colon \mathbb{Q}[x] \to \mathbb{C}$ be the function which is given by $\phi(f(x)) = f(\sqrt{2})$. All of us know that this function is a ring homomorphism; you do not have to show me a proof. What is the kernel of ϕ? Prove your answer.**

We show that $\boxed{\ker\phi = (x^2-2)}$.

$(\supseteq)$. Every element of (x^2-2) has the form $(x^2-2)g(x)$ for some $g(x)$ in $\mathbb{Q}[x]$. We see that

$$\phi((x^2-2)g(x)) = ((\sqrt{2})^2-2)g(\sqrt{2}) = 0.$$

Thus, $(x^2-2)g(x)$ is in the kernel of ϕ.

$(\subseteq)$. Let $f(x)$ be in $\ker\phi$. Divide x^2-2 into $f(x)$ to get $f(x) = (x^2-2)g(x)+ax+b$ for some polynomials $g(x)$ and $ax+b$ in $\mathbb{Q}[x]$. Apply ϕ to both sides to see that $0 = a\sqrt{2}+b$. The number $\sqrt{2}$ is not a rational number; hence, $a=b=0$ and $f(x) \in (x^2+2)$.

6. **(8 points) Let $\phi\colon R \to S$ be a ring homomorphism.**
   (a) Prove that $\phi(0)=0$.
   (b) Prove that ϕ is one-to-one if and only if $\ker\phi = \{0\}$.

   (a) We know that ϕ is a homomorphism; hence, $\phi(0) = \phi(0+0) = \phi(0)+\phi(0)$. We know that every element of S (in particular $\phi(0)$) has an additive inverse in S. Add the additive inverse of $\phi(0)$ to both sides to conclude that $0 = \phi(0)$.

   (b) $(\Rightarrow)$ If ϕ is one-to-one, then only one element of R is sent to the zero element of S. We saw in part (a) that 0 is sent to 0. It follows that 0 is the only element of R which is sent to zero and $\ker\phi = \{0\}$.

$(\Leftarrow)$ We assume $\ker\phi = \{0\}$. Suppose r_1 and r_2 are in R with $\phi(r_1) = \phi(r_2)$. We see that

$$\phi(r_1-r_2) = \phi(r_1)-\phi(r_2) = 0;$$

thus, $r_1 - r_2 \in \ker\phi = \{0\}$, and $r_1 = r_2$. We conclude that ϕ is one-to-one.

7. **(7 points) Let M be an ideal of the ring R. Suppose that $M \neq R$ and that R is the only ideal of R which properly contains M. Prove that $\frac{R}{M}$ is a field.**

We need only show that each non-zero element of $\frac{R}{M}$ has a multiplicative inverse in $\frac{R}{M}$. Pick a non-zero element of $\frac{R}{M}$. This element has the form $\bar{a}$ where a is an element of R which is not an element of M. We must show that the element $\bar{a}$ of $\frac{R}{M}$ has an inverse in $\frac{R}{M}$.

Let (M,a) denote the smallest ideal of R which contains M and a. Observe that $(M,a) = \{m+ra \mid m \in M \text{ and } r \in R\}$. The hypothesis ensures us that $(M,a) = R$. In other words, there exist elements $m \in M$ and $r \in R$ with $1 = m + ra$. We conclude that $\bar{r}$ is the inverse of $\bar{a}$ in $\frac{R}{M}$.