Math 546 Summer 2002 Final Exam

There are 20 problems on 10 pages. Each problem is worth 5 points.

1. Define "group isomorphism". Use complete sentences.
The function φ from the group G_1 to the group G_2 is a group isomorphism if φ is one-to-one, onto, and $\varphi(xy) = \varphi(x)\varphi(y)$ for all x and y in G_1.

2. Define "normal subgroup". Use complete sentences.
The subgroup N of the group G is a normal subgroup if gng^{-1} is in N for every $g \in G$ and $n \in N$.

3. Define "centralizer". Use complete sentences.
Let x be an element of the group G. The centralizer of x in G is the set of elements in G which commute with x.

4. Define "center". Use complete sentences.
The center of the group G is the set of elements in G which commute with every element of G.

5. Define "cyclic group". Use complete sentences.
The group G is a cyclic group if there exists an element g in G with the property that every element of G is equal to g to some power.

6. State and PROVE Lagrange's Theorem.
Statement. If H is a subgroup of the finite group G, then the order of H divides the order of G.
Proof. For each element $g \in G$, consider the right coset $Hg = \{hg \mid h \in H\}$. We will prove
(a) Every element of G is in exactly one right coset of H in G.
(b) Every right coset of H in G has the same number of elements as H.

Once we have established (a) and (b), then we will know that the number of elements in G is equal to the number of cosets times the number of elements in each coset. In other words, $|G| = r|H|$, where r is the number of cosets, $|G|$ is the order of G, and $|H|$ is the order of H.
Proof of (a). Let g be an arbitrary element of G. We know that g is in the right coset Hg. Suppose that g is also in the right coset Hg', for some $g' \in G$. We will show that the cosets Hg and Hg' are equal. The hypothesis $g \in Hg'$ ensures that there exists an element h' of H, with

$$g = h'g'. \qquad (*)$$

We first show that $Hg \subseteq Hg'$. Take a typical element hg of Hg, for some $h \in H$. We see from (*) that $hg = hh'g'$, and we know that hh' is in H, because H is a group. Thus, $hg \in Hg'$.

Now we show that $Hg' \subseteq Hg$. Take a typical element hg' of Hg', for some $h \in H$. We see from (*) that $hg' = h(h')^{-1}g$. Once again, we know that $h(h')^{-1}$ is an element of H, because H is a group. It follows that $hg' \in Hg$.

We have shown that $Hg \subseteq Hg'$ and $Hg' \subseteq Hg$. We conclude that $Hg' = Hg$; and therefore, every element of G is in exactly one right coset of H in G.
Proof of (b). Let g be an arbitrary element of G. We establish a one-to-one correspondence between the sets H and Hg. Define $\varphi : \mathrm{H} \to Hg$, by $\varphi(h) = hg$ for each h in H. Observe that φ is onto. Indeed, if x is an arbitrary element of the coset Hg, then $x = hg$ for some h in H, and φ of this h is equal to x. It is also clear that φ is one-to-one. Indeed, if h and h' are elements of H, with $\varphi(h) = \varphi(h')$, then $hg = h'g$ in the group G. We may multiply by g^{-1} to conclude that $h = h'$.

The one-to-one correspondence φ from H to Hg shows that H and Hg have the same number of elements.

The proof is complete.

7. PROVE that every subgroup of $(\mathbb{Z}, +)$ is cyclic. I do NOT want you to prove a more general statement. I want you to prove the statement I have written. I want you to use notation which is appropriate to the **additive** group $\mathbb{Z}$.

Let H be a subgroup of $\mathbb{Z}$. If H consists of only zero, then H is cyclic. Henceforth, we assume that H contains more elements than just 0. As soon as some integer n is in H, then the inverse of n, which is $-n$, is also in H. Consequently, we know that H contains some positive integer. Let h_0 be the smallest positive integer in H. I will prove that $H = <h_0>$. It is clear that $<h_0> \subseteq H$. We need only show that $H \subseteq <h_0>$. Let h be an arbitrary element of H. Divide h_0 into h. We see that h_0 goes into h, n times for some integer n, with a remainder r for some integer r, with $0 \le r \le h_0 - 1$. That is, $h = nh_0 + r$. It follows that $r = h - nh_0$, which is an element of H because H is a group. On the other hand, r is non-negative and less than h_0. Our choice of h_0 tells us that r must be zero; hence, $h = nh_0$ and $h \in <h_0>$. We conclude that $H = <h_0>$; and therefore, H is a cyclic group.

8. Write down four groups. Each group is to have eight elements. None of the groups is to be isomorphic to any of the others. Explain thoroughly.

Consider D_4, $\mathbb{Z}_8$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. I have listed four groups. The group D_4 is the only non-abelian group on my list, so it is not isomorphic to any of the other groups. The group $\mathbb{Z}_8$ is the only cyclic group on my list, so it is not isomorphic to any of the other groups. The groups $\mathbb{Z}_4 \times \mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ are not isomorphic because $\mathbb{Z}_4 \times \mathbb{Z}_2$ contains some elements of order 4, but every element of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ has order 2 or 1.

9. Let $\mathbb{R}^{\mathrm{pos}}$ represent the group of positive real numbers under multiplication. Prove that the groups $(\mathbb{R}, +)$ and $(\mathbb{R}^{\mathrm{pos}}, \times)$ are isomorphic.

Define $\varphi \colon \mathbb{R} \to \mathbb{R}^{\mathrm{pos}}$ by $\varphi(r) = e^r$. Notice that

$$\varphi(r + s) = e^{r+s} = e^r e^s = \varphi(r)\varphi(s);$$

thus, φ is a homomorphism. We next show that φ is onto. Let g be an arbitrary element of $\mathbb{R}^{\mathrm{pos}}$. Notice that $\ln g$ is in $\mathbb{R}$ and $\varphi(\ln g) = e^{\ln g} = g$.

Now we show that f is one-to-one. Take r and s in $\mathbb{R}$ with $\varphi(r) = \varphi(s)$. That is, $e^r = e^s$. Take the natural logarithm of each side to conclude that $r = s$.

10. Give an example of a subgroup of S_4 which has six elements. Explain.
The group $S_3 = \{(1), (12), (13), (23), (123), (132)\}$ is a subgroup of S_4.

11. Give an example of a subgroup of $(\mathbb{C} \setminus \{0\}, \times)$ which has six elements. Explain.
The group $U_6 = \{1, u, u^2, u^3, u^4, u^5\}$, for $u = \cos\frac{2\pi}{6} + \imath \sin\frac{2\pi}{6}$ is a subgroup of $(\mathbb{C} \setminus \{0\}, \times)$.

12. How many elements of S_5 have order 2? Explain.
There are $\binom{5}{2} = 10$ transpositions in S_5. There are 5 times 3 elements of S_5 of the form $(ij)(k\ell)$ with i, j, k, ℓ all distinct. Thus S_5 has $\boxed{25}$ elements of order 2.

13. Express the permutation $(6,9)(1,2)(4,9,7)(4,8)(1,2,3)$ as a product of disjoint cycles. This permutation is an element of the group S_9.
This permutation is equal to $\boxed{(2,3)(4,8,6,9,7)}$.

14. Let $(G, *)$ be an abelian group. Let S be the set of all elements g in G which satisfy the equation $g * g * g = \text{id}$. Prove that S is a subgroup of G.
We show that S is closed. Take g and h from S. We know that $g * g * g =$ id and $h * h * h =$ id. We must show that gh is in S. The group G is abelian; hence,

$$gh * gh * gh = ggg * hhh = \textbf{id}.$$

It follows that $gh \in S$. Take $g \in S$. We must show that the inverse of g is also in S. The defining equation for S tells us that g's inverse is $g * g$. We already have shown that S is closed under $*$. Thus, $g * g$, which is g's inverse, is also in S. Of course, the identity element of G cubes to id, so id is in S.

15. Let $(G, *)$ be the group $(\mathbb{Z}_3 \times \mathbb{Z}_6, +)$. **LIST** all of the elements of $(G, *)$ which satisfy the equation $g * g * g = \text{id}$. No explanation is needed.
The elements g of G with $g * g * g =$ id are

$$(0,0),\ (0,2),\ (0,4),\ (1,0),\ (1,2),\ (1,4),\ (2,0),\ (2,2),\ (2,4).$$

16. Is $(\mathbb{Z}_{15}^{\times}, \times)$ a cyclic group? Explain.
NO! The group consists of 8 elements. We see that $2^4 = 1$, $4^2 = 1$, $7^4 = (7^2)^2 = (4)^2 = 1$, $8^4 = (-7)^4 = 1$, $(11)^2 = (-4)^2 = 1$, $(13)^4 = (-2)^4 = 1$, $(14)^2 = (-1)^2 = 1$. Every element of this group has order 4 or less.

17. Is $(\mathbb{Z}_2 \times \mathbb{Z}_3, +)$ a cyclic group? Explain.

YES! The group is generated by $(1,1)$.

18. The group D_4 has three distinct subgroups of order 4. List the elements of each of these subgroups. (I do not need to see any details.)

The subgroups are $\{\mathbf{id}, \rho, \rho^2, \rho^3\}$, $\{\mathbf{id}, \sigma, \sigma\rho^2, \rho^2\}$, and $\{\mathbf{id}, \sigma\rho, \sigma\rho^2, \rho^3\}$.

19. The subgroup $V = \{(1), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$ of the group S_4 is normal. (You do not have to prove this.) Find an element of the factor group $\frac{S_4}{V}$ which has order 3. Explain.

The coset $V(123)$ has order 3 because when I square this coset I get $V(132)$, which is not the identity element of the factor group and when I cube this coset I get V, which is the identity element of the factor group.

20. Let $\mathbb{R}^{\text{pos}}$ represent the group of positive real numbers under multiplication and let U be the unit circle. If z is the complex number $a + b\imath$, then the modulus $|z|$ of z is equal to $\sqrt{a^2+b^2}$. Define $\varphi\colon \frac{\mathbb{C}\setminus\{0\}}{U} \to \mathbb{R}^{\text{pos}}$ by $\varphi(Uz) = |z|$, for each coset Uz of $\frac{\mathbb{C}\setminus\{0\}}{U}$. Prove that φ is a group isomorphism.

First, we show that φ is a well defined function. Suppose that the cosets Uz and Uw are equal, for some non-zero complex numbers z and w. In this case, $z = uw$ for some $u \in U$. We saw on the second day of class (or we can calculate again) that

$$(**) \qquad |z_1 z_2| = |z_1| \cdot |z_2|$$

for all z_1 and z_2 in $\mathbb{C}$. In particular, $|z| = |uw| = |u| \cdot |w| = |w|$; so, φ carries both names, Uz and Uw, to the same number $|z|$ in $\mathbb{R}^{\text{pos}}$.

Now we show that φ is a homomorphism. Take z_1 and z_2 in $\mathbb{C} \setminus \{0\}$. Observe, using (**), that

$$\varphi(Uz_1 \cdot Uz_2) = \varphi(Uz_1 z_2) = |z_1 z_2| = |z_1| \cdot |z_2| = \varphi(Uz_1) \cdot \varphi(Uz_2).$$

Now we show that φ is onto. Take $r \in \mathbb{R}^{\text{pos}}$. Notice that the coset Ur is in $\frac{\mathbb{C}\setminus\{0\}}{U}$ and $\varphi(Ur) = |r| = r$.

Finally, we show that φ is one-to-one. Suppose that z_1 and z_2 are in $\mathbb{C} \setminus \{0\}$ with $\varphi(Uz_1) = \varphi(Uz_2)$. It follows that $|z_1| = |z_2|$; hence, $\frac{z_1}{z_2}$ has modulus 1, and is equal to an element u of U. We see that $z_1 = \frac{z_1}{z_2} z_2 = uz_2$. We conclude that the cosets Uz_1 and Uz_2 are equal.