PRINT Your Name:___________________________________

Quiz for April 8, 2004

Let $G = \{x \in \mathbb{R} \mid x > 0 \text{ and } x \neq 1\}$. Define $*$ on G by $a * b = a^{\ln b}$. Prove that G is isomorphic to $(\mathbb{R} \setminus \{0\}, \times)$.

ANSWER: Define $\phi\colon \mathbb{R} \setminus \{0\} \to G$ by $\phi(r) = e^r$. Notice that ϕ is a function and $\phi(r) \in G$ for all $r \in \mathbb{R}$ with $r \neq 0$. In other words, e^r is positive and not equal to 1. It is clear that ϕ is one-to-one. If r and s are in $\mathbb{R} \setminus \{0\}$ and $\phi(r) = \phi(s)$, then $e^r = e^s$. Apply $\ln$ to both sides to see that $r = s$. It is clear that ϕ is onto. If $g \in G$, then let $r = \ln g$. Notice that r is a real number and $r \neq 0$ because $g \neq 1$. Also $\phi(r) = e^r = e^{\ln g} = g$. Now we check that ϕ is a homomorphism. Take r and s from $\mathbb{R} \setminus \{0\}$. We see that

$$\phi(r) * \phi(s) = e^r * e^s = (e^r)^{\ln e^s} = (e^r)^s = e^{rs} = \phi(rs).$$