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Quiz for March 4, 2004

Let G be a group with a and b in G. Assume that $o(a)$ and $o(b)$ are finite and relatively prime, and that $ab = ba$. Prove that $o(ab) = o(a)o(b)$.

ANSWER: Let $\ell = o(a)$, $m = o(b)$, and $n = o(ab)$. Since ℓ, m and n all are positive integers, it suffices to prove that $n|\ell m$ and $\ell m|n$.

$n|\ell m$: The elements a and b commute; hence,

$$(ab)^{\ell m} = a^{\ell m}b^{\ell m} = (a^\ell)^m(b^m)^\ell = \text{id}.$$

So, $(ab)^{\ell m}$ is the identity. It follows that n, which is the order of ab, must divide ℓm.

$\ell m|n$: Observe that

$$\text{id} = ((ab)^n)^\ell = (a^\ell)^n b^{n\ell} = b^{n\ell}.$$

The order of b is m; thus, $m|n\ell$. The integers m and ℓ are relatively prime; thus, $m|n$.
In a similar manner, we see that

$$\text{id} = ((ab)^n)^m = a^{mn}(b^m)^n = a^{mn}.$$

The order of a is ℓ; thus, $\ell|mn$. The integers ℓ and m are relatively prime; so, $\ell|n$.
Finally, we notice that $m|n$ and $\ell|n$, with ℓ and m relatively prime. It follows that $m\ell|n$, and the proof is complete.