PRINT Your Name:__________________________________

Quiz for August 25, 2011

Let $S = \mathbb{R} \setminus \{-1\}$. Define $*$ on S by $a * b = a + b + ab$. Prove that $(S, *)$ is a group.

ANSWER:

Closure: Take a, b from S. We must show that $a * b$ is in S. Well, $a * b = a + b + ab$, which is clearly a real number. We must check that $a + b + ab$ is not equal to -1. If $a + b + ab$ were equal to -1, then $a + b + ab = -1$; so, $1 + a + b + ab = 0$; that is, $(1 + a)(1 + b) = 0$; so $a = -1$ or $b = -1$. On the other hand, a and b are in S; so neither a nor b is -1. We conclude that $a + b + ab \neq -1$; therefore, $a + b + ab \in S$

Associativity: Take a, b, and c from S. Observe that

$$a*(b*c) = a*(b+c+bc) = a+(b+c+bc)+a(b+c+bc) = a+b+c+ab+ac+bc+abc.$$

On the other hand,

$$(a*b)*c = (a+b+ab)*c = (a+b+ab)+c+(a+b+ab)c = a+b+c+ab+ac+bc+abc.$$

We see that $a * (b * c) = (a * b) * c$.

Identity: The number 0 is the identity element of S because $a * 0 = a + 0 + a(0) = a$ and $0 * a = 0 + a + 0(a) = a$ for all $a \in S$.

Inverses: Take $a \in S$. The inverse of a is $\frac{-a}{1+a}$ because

$$a * \frac{-a}{1+a} = a + \frac{-a}{1+a} + a\frac{-a}{1+a} = a + \frac{-a(1+a)}{1+a} = a - a = 0.$$

The operation $*$ is commutative; so, $\frac{-a}{1+a} * a$ is also equal to 0. Notice, also, that $\frac{-a}{1+a} \in S$ because $\frac{-a}{1+a}$ is a real number (since $a \neq -1$) and $\frac{-a}{1+a}$ is not equal to -1; because if $\frac{-a}{1+a}$ were equal to -1, then $\frac{-a}{1+a} = -1$, so $-a = -1 - a$; that is, $0 = -1$.