Homework November 15, 2004

section 2.3, page 75 — 1a, 3 (for $\sigma\tau$), 4, 5a, 6 (list all of the elements of S_4, use cycle notation), 7.

section 3.2, page 101 — 6.

1. Let G be a group. For each element a in G, let λ_a be the function from G to G, which is defined by $\lambda_a(g) = ag$.
   (a) Prove that $\lambda_a\colon G \to G$ is one-to-one and onto. (Once you have completed this part of the problem, then you have shown that each λ_a is an element of $\mathrm{Sym}(G)$.)
   (b) Prove that the function $\Lambda\colon G \to \mathrm{Sym}(G)$, which is given by $\Lambda(a) = \lambda_a$, is a group homomorphism. (If a and b are elements of G then you must show that the FUNCTIONS $\Lambda(ab)$ and $\Lambda_a \circ \Lambda_b$ are equal. One usually shows that two functions are equal by showing that they do the same thing to each element of the domain.)
   (c) Prove that Λ is one-to-one. (Once you have completed this part of the problem, then you have proven that G is isomorphic to a subgroup of the permuatation group $\mathrm{Sym}(G)$. This is called Cayley's Theorem.)

2. Let G be the group D_3 with elements $\mathrm{id}, \rho, \rho^2, \sigma, \sigma\rho, \sigma\rho^2$. Compute the homomorphism $\Lambda\colon D_3 \to \mathrm{Sym}(D_3)$ as described in 1. It is natural to think of $\mathrm{Sym}(D_3)$ as S_6 where the elements of D_3 are identified with $\{1,2,3,4,5,6\}$ by $1 \leftrightarrow \mathrm{id}$, $2 \leftrightarrow \rho$, $\ldots$, $6 \leftrightarrow \sigma\rho^2$, using the above order for the elements of D_3. For each g in D_3, find the permutaion $\Lambda(g)$ in S_6.

3. Let G be the group $\mathbb{Z}_6$ with elements $1,2,3,4,5,0$. Compute the homomorphism $\Lambda\colon \mathbb{Z}_6 \to \mathrm{Sym}(\mathbb{Z}_6)$ as described in 1. It is natural to think of $\mathrm{Sym}(\mathbb{Z}_6)$ as S_6 where the elements of $\mathbb{Z}_6$ are identified with $\{1,2,3,4,5,6\}$ by $1 \leftrightarrow 1$, $2 \leftrightarrow 2$, $\ldots$, $6 \leftrightarrow 0$, using the above order for the elements of $\mathbb{Z}_6$. For each g in $\mathbb{Z}_6$, find the permutaion $\Lambda(g)$ in S_6.

4. The goal of this exercise is to prove that it makes sense to say "even permutation" or odd permutation". Your book offers two proofs: one on 74–75 and one on 133-134. This exercise outlines the argument given on 133-134. Let $\mathbb{Z}[x_1,\ldots,x_n]$ be the collection of polynomials in n variables with integer coefficients. Notice that each element of S_n gives a permutation of $\mathbb{Z}[x_1,\ldots,x_n]$ by $\sigma(f(x_1,\ldots,x_n))$ is equal to $f(x_{\sigma(1)},\ldots,x_{\sigma(n)})$. Let Δ_n be the polynomial
$$\Delta_n = \prod_{1\le i<j\le n} (x_i - x_j)$$
in $\mathbb{Z}[x_1,\ldots,x_n]$.
   (a) Let σ and τ be in S_n and $f \in \mathbb{Z}[x_1,\ldots,x_n]$. Observe that $(\sigma\circ\tau)(f) = \sigma(\tau(f))$.
   (b) Observe that if $\sigma \in S_n$, then $\sigma(\Delta_n)$ is equal to Δ_n or $-\Delta_n$.

(c) Prove that if $\sigma \in S_n$ is a transposition, then $\sigma(\Delta_n) = -\Delta_n$.
(d) Prove that $\phi\colon S_n \to \{1, -1\}$, given by $\phi(\sigma) = \frac{\Delta_n}{\sigma(\Delta_n)}$ is a group homomorphism.
(e) What is the kernel of ϕ?

5. Let r and s be distinct elements of $\{1, 2, \ldots, n\}$, with $n \geq 3$. Prove that A_n is generated by the 3-cycles $\{(rsk) \mid 1 \leq k \leq n, k \neq r, s\}$.

6. Let N be a normal subgroup of A_n for some $n \geq 3$. Suppose that N contains a 3-cycle. Prove that $N = A_n$.

7. Prove that A_4 does not have any subgroups of order 6.

8. Fix $n \geq 5$. The purpose of this problem is to prove that A_n is a simple group, that is, that A_n does not contain any normal subgroups other than $\{\text{id}\}$ and A_n. (This result is a key step in the proof of Galois's Theorem that there does not exist an algebraic formula which expresses the roots of an arbitrary fifth degree polynomial in terms of the coefficients.) We let N be a normal subgroup of A_n, with $N \neq \{\text{id}\}$. We will prove that N must equal A_n.

(a) If N contains a three cycle, then prove that $N = A_n$.
(b) Let $\sigma = (a_1, a_2, \ldots, a_r)\tau$ be a decomposition into disjoint cycles with $r \geq 4$. Suppose that σ is in N. Let $\delta = (a_1, a_2, a_3)$. Calculate $\sigma^{-1}(\delta\sigma\delta^{-1})$. Prove that $N = A_n$.
(c) Let $\sigma = (a_1, a_2, a_3)(a_4, a_5, a_6)\tau$ be a decomposition into disjoint cycles. Suppose that σ is in N. Let $\delta = (a_1, a_2, a_4)$. Calculate $\sigma^{-1}(\delta\sigma\delta^{-1})$. Prove that $N = A_n$.
(d) Let $\sigma = (a_1, a_2, a_3)\tau$ be a decomposition into disjoint cycles, where τ is a product of disjoint transpositions. Suppose that σ is in N. Calculate σ^2. Prove $N = A_n$.
(e) Let $\sigma = (a_1, a_2)(a_3, a_4)\tau$ be a decomposition into disjoint cycles, where τ is a product of disjoint transpositions. Suppose that σ is in N. Let $\delta = (a_1, a_2, a_3)$. Calculate $\sigma^{-1}(\delta\sigma\delta^{-1})$. Let b be an element of $\{1, \ldots, n\}$ with b distinct from a_1, a_2, a_3. Let $\xi = (a_1, a_3, b)$ and $\zeta = (a_1, a_3)(a_2, a_4)$. Calculate $\zeta(\xi\zeta\xi^{-1})$. Prove $N = A_n$.
(f) Make sure that we have covered all of the possibilities! Conclude that if $n \geq 5$, then A_n is a simple group.