Homework October 1, 2004

1. Suppose that H is a subgroup of the group G and ghg^{-1} is in H for all $g \in G$ and $h \in H$.

(a) Let h_1 be an arbitrary element of H and g be an arbitrary element of G. Prove that there exists an element h of H with $h_1 = ghg^{-1}$. (It is possible to give a proof which works for infinite groups as well as finite groups.)

(b) Let a, b, c, and d be elements of G with $aH = bH$ and $cH = dH$. Prove that $acH = bdH$. (This is only a tiny extension of homework number 3 from September 29.)

(c) Let S be the set of cosets $S = \{aH \mid a \in G\}$ of H in G. Part (b) shows that the operation on S given by $(aH) * (bH) = abH$ is a well-defined function. Prove that S is a group. (If you are looking for this somewhere, S is usually written as $\frac{G}{H}$ and S is called the "quotient group of G mod H", or the "factor group of G mod H". BY THE WAY: S is not a subset of anything; we have to verify all of the axioms for group. Fortunately, this is very easy.)

2.

(a) If G is an abelian group and H is a subgroup of G, then prove that ghg^{-1} is in H for all $g \in G$ and $h \in H$.

(b) If G is a finite group with $2n$ elements and H is a subgroup of G with n elements, then prove that ghg^{-1} is in H for all $g \in G$ and $h \in H$.

(b) If G is a group and H is a subgroup of the center of G, then prove that ghg^{-1} is in H for all $g \in G$ and $h \in H$.

3. Work out some examples of $\frac{G}{H}$ as described in problem 1c.

(a) Let $G = D_4$ and $H = <\rho>$. Problem 2c tells us that it is legal to create $\frac{G}{H}$. What is this group? How many elements does it have? What is the multiplication table? Do you believe that this multiplication makes sense?

(b) Let $G = D_4$ and $H = <\rho^2>$. Problem 2b tells us that it is legal to create $\frac{G}{H}$. What is this group? How many elements does it have? What is the multiplication table? Do you believe that this multiplication makes sense?

(c) Let $G = \mathbb{Z}$ and $H = 5\mathbb{Z}$. Problem 2a tells us that it is legal to create $\frac{G}{H}$. What is this group? How many elements does it have? What is the addition table? Do you believe that this addition makes sense? (Notice that the elements of this $\frac{G}{H}$ look like $a+H$ because the operation in G is called $+$. Furthermore, the operation in $\frac{G}{H}$ is also called $+$; that is, $(a+H)+(b+H) = a+b+H$.)