Math 546, Final Exam, Spring, 2001

PRINT Your Name:________________________________

Get your course grade from **TIPS/VIP** late on Tuesday or later.

There are 20 problems on 8 pages. The exam is worth a total of 150 points. Problems 1 through 10 are worth eight points each. Problems 11 through 20 are worth 7 points each.

1. DEFINE *group.*

2. DEFINE *cyclic group.*

3. DEFINE the *center* of a group.

4. DEFINE *normal subgroup.*

5. STATE Lagrange's Theorem.

6. STATE the lemma from number theory about linear combinations and greatest common divisors.

7. STATE the "Chinese Remainder Theorem" about the group $\mathbb{Z}_n \times \mathbb{Z}_m$ and the group $\mathbb{Z}_{nm}$.

8. STATE the lemma about the order of the element ab in terms of the order of a and the order of b .

9. STATE the two results about the subgroups of a cyclic group.

10. Pick one of the statements from problems 5 through 9. Tell me which statement you have chosen. PROVE the statement.

11. Pick a second statement from problems 5 through 9. Tell me which statement you have chosen. PROVE the statement.

12. What is the order of the element $(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}, \rho)$ in the group $U_6 \times D_4$? Explain your answer.

13. Let G be the group $\mathbb{Z}_4 \times \mathbb{Z}_{10}$. Let N be the subgroup $<(2,2)>$ of G . What is the order of the element $(1,2)+N$ in the group $\frac{G}{N}$? Explain your answer.

14. Let $(\mathbb{R}^{\text{pos}}, \times)$ represent the group of positive real numbers under multiplication. Does $(\mathbb{R}^{\text{pos}}, \times)$ contain any non-cyclic subgroups? If not, explain why not. If so, exhibit such a subgroup and explain why the subgroup is not cyclic.

15. TRUE or FALSE. (If true, PROVE it. If false, give a COUNTER EXAMPLE.) All groups of order 6 are isomorphic.

16. TRUE or FALSE. (If true, PROVE it. If false, give a COUNTER EXAMPLE.) All groups of order 7 are isomorphic.

17. How many permutations in S_6 have order 4. Explain your answer.

18. Let G be the group of non-zero complex numbers under multiplication. Let G' be the group of non-zero 2×2 matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, with real entries, under multiplication. Consider the function $\varphi \colon G \to G'$, which is given by $\varphi(a+bi) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$. Prove that φ is a group isomorphism.

19. Let K and N be subgroups of the group G. Let

$$S = \{kn \mid k \in K \text{ and } n \in N\}.$$

If N is a normal subgroup of G, then prove that S is a subgroup of G.

20. The subgroup $N = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\}$ of the group S_4 is normal. The factor group $\frac{S_4}{N}$ is isomorphic to which familiar group? Explain your answer.