3. Define "centralizer". Use complete sentences.

Let g be an element in the group G. The <u>centralizer</u> of g in G is

$$\{x \in G \mid gx = xg\}.$$

4. Let A be the element $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ in the group $G = \mathrm{GL}_2(\mathbb{R})$. Find the centralizer of A in G.

$$C(A) = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \middle| \; a, b \in \mathbb{R} \text{ with } a^2 \neq b^2 \right\}$$

Pf
It is clear that $C(A)$ is a subset of G.
We first show every such matrix commutes with A

$$A \begin{bmatrix} a & b \\ b & a \end{bmatrix} = \begin{bmatrix} a+2b & 2a+b \\ b+2a & 2b+a \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix} A = \begin{bmatrix} a+2b & 2a+b \\ b+2a & 2b+a \end{bmatrix}$$

These two matrices are equal

so $\left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \middle| \; a, b \in \mathbb{R} \text{ with } a^2 \neq b^2 \right\} \subseteq C(A)$

Now take an arbitrary element of $C(A)$
This element looks like $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $ad-bc \neq 0$.

$$A \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+2c & 2d+b \\ c+2a & 2b+d \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} A = \begin{bmatrix} a+2b & 2a+b \\ c+2d & 2c+d \end{bmatrix}$$

So
$$\begin{aligned} a+2c &= a+2b \\ 2d+b &= 2a+b \\ c+2a &= c+2d \\ 2b+d &= 2c+d \end{aligned}$$

The top equation tells me $c = b$

The second equation tells me $d = a$

So every element of $C(A)$ looks like $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ with $a^2 - b^2 \neq 0$.