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10. Let $T = \mathbb{R} \setminus \{1\}$. Define $*$ on T by $a * b = ab - a - b + 2$. Prove that $(T, *)$ is isomorphic to $(\mathbb{R} \setminus \{0\}, \times)$.

Define $\varphi : \mathbb{R} \setminus \{0\} \longrightarrow T$ by $\varphi(r) = r+1$.

φ is 1-1: If $r, s \in \mathbb{R}$ with $\varphi(r) = \varphi(s)$ then $r+1 = s+1$ so $r = s$

φ is onto: If $t \in T$ then $t \neq 1$ and $t \in \mathbb{R}$ so $t-1 \neq 0$ and $t-1 \in \mathbb{R}$

Also $\varphi(t-1) = t$

φ respects the operations

Take $r, s \in \mathbb{R} \setminus \{0\}$

$\varphi(rs) = rs+1$

$$\varphi(r) * \varphi(s) = (r+1) * (s+1) = (r+1)(s+1) - (r+1) - (s+1) + 2$$

$$= rs + \cancel{r} + \cancel{s} + 1 - \cancel{r} - 1 - \cancel{s} - 1 + 2$$

$$= rs + 1$$

so $\varphi(rs) = \varphi(r) * \varphi(s)$.