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6. Find all of the subgroups of $U_9 = \{z \in \mathbb{C} \mid z^9 = 1\}$. Explain why you are certain that you have found all of the subgroups.

$U_9 = \langle u \rangle$ where $u = \cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}$.
$|U_9| = 9$ Lagrange's Theorem tells me that every subgroup other than $\langle id \rangle$ and $U_9$ has order 3. Lagrange's theorem also tells me that every group of order 3 is cyclic.

The subgroups of $U_9$ are
$\langle u \rangle = \langle u^2 \rangle = \langle u^4 \rangle = \langle u^5 \rangle = \langle u^7 \rangle = \langle u^8 \rangle = U_9$
$\langle u^3 \rangle = \langle u^6 \rangle = \{u^3, u^6, 1\}$
and $\{1\}$

$u$ and $u^8$ are inverses of one another so $\langle u \rangle = \langle u^8 \rangle$
$u^2$ and $u^7$ are inverses of one another so $\langle u^2 \rangle = \langle u^7 \rangle$
$u^4$ and $u^5$ are inverses of one another so $\langle u^4 \rangle = \langle u^5 \rangle$.

Lagrange tells me that $\langle u^2 \rangle = U_9$ because
$\langle u^2 \rangle \supseteq \{1, u^2, u^4, u^6\}$
so $\langle u^2 \rangle$ is too big to be a group of ord.
similarly
$\langle u^4 \rangle \supseteq \{1, u^4, u^8, u^3\}$
so $|\langle u^4 \rangle| > 3$ so $\langle u^4 \rangle = U_9$.

7. Let $H$ be a subgroup of the group $G$. Let $a$ be a fixed element of $G$ and let

$$K = \{aha^{-1} \mid h \in H\}.$$

Prove that $K$ is a subgroup of $G$.

✓ id $= a\,id\,a^{-1}$ and id $\in H$ so id $\in K$

✓ closure Take $x = aha^{-1}$ and $y = ah'a^{-1}$ in $K$ for $h, h' \in H$
so $xy = aha^{-1}ah'a^{-1} = ahh'a^{-1}$ and this is in $K$ because $hh' \in H$.

✓ inverses Take $x = aha^{-1} \in K$ for $h \in H$
$x^{-1} = ah^{-1}a$ and this is in $K$ because $h^{-1} \in H$.