Math 546, Final Exam, Spring 2004

PRINT Your Name:________________________________

There are 17 problems on 6 pages. The exam is worth 100 points.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**. Otherwise, get your course grade from VIP.

I will post the solutions on my website on **Wednesday**.

1. (5 points) Define "centralizer". Use complete sentences.

2. (5 points) Define "normal subgroup". Use complete sentences.

3. (6 points) (Yes or No. If yes, PROVE it. If no, give a COUNTEREXAMPLE.) Let a and b be elements of finite order in the group G. Does ab have to have finite order?

4. (6 points) Recall that each element of $\mathbb{C}$ is a point on the complex plane. Notice that $(\mathbb{R}^{\text{pos}}, \times)$ is a subgroup of $(\mathbb{C} \setminus \{0\}, \times)$. Give a geometric description of the left cosets of $(\mathbb{R}^{\text{pos}}, \times)$ in $(\mathbb{C} \setminus \{0\}, \times)$.

5. (6 points) (Yes or No. If yes, PROVE it. If no, give a COUNTEREXAMPLE.) Let a be a fixed element of the group G. Consider the function $\rho_a \colon G \to G$, which is given by $\rho_a(g) = ga$, for all g in G. Is ρ_a onto?

6. (6 points) (Yes or No. If yes, PROVE it. If no, give a COUNTEREXAMPLE.) Let a be a fixed element of the group G. Consider the function $\rho_a \colon G \to G$, which is given by $\rho_a(g) = ga$, for all g in G. Is ρ_a a homomorphism?

7. (6 points) (Yes or No. If yes, PROVE it. If no, give a COUNTEREXAMPLE.) Is $\varphi \colon \mathbb{Z}_{10} \to \mathbb{Z}_5$, which is given by $\varphi([n]_{10}) = [n]_5$, a function?

8. (6 points) (Yes or No. If yes, PROVE it. If no, give a COUNTEREXAMPLE.) Is $\varphi \colon \mathbb{Z}_5 \to \mathbb{Z}_{10}$, which is given by $\varphi([n]_5) = [n]_{10}$, a function?

9. (6 points) Let N be a normal subgroup of the group G, and let $\frac{G}{N}$ be the set of left cosets of N in G. Prove that $\varphi \colon \frac{G}{N} \times \frac{G}{N} \to \frac{G}{N}$, which is given by
$$\varphi(aN, bN) = abN,$$
is a function.

10. (6 points) (Yes or No. If yes, PROVE it. If no, give a COUNTEREXAMPLE.) Let $f \colon \mathbb{Z} \to \mathbb{Z}$ be a one-to-one and onto function. Suppose $B \subseteq \mathbb{Z}$ with $f(B) \subseteq B$. Is $f(B) = B$?

11. (6 points) What is the order of $([2]_6, [2]_4) + <([3]_6, [2]_4)>$ in $\frac{\mathbb{Z}_6 \times \mathbb{Z}_4}{<([3]_6,[2]_4)>}$? Explain.

12. (6 points) Let H be a non-zero subgroup of $\mathbb{Z}$. Prove that H is cyclic.

13. (6 points) Let d be the greatest common divisor of the integers n and m. Prove that there exist integers r and s with $rn + sm = d$.

14. (6 points) List 6 subgroups of the Dihedral group D_4. No explanation is needed.

15. (6 points) Prove that $(\mathbb{R}, +)$ is isomorphic to $(\mathbb{R}^{\text{pos}}, \times)$.

16. (6 points) Consider $(\mathbb{Z}, *)$, where $n * m = n + m + 1$ for all integers n and m. Is $(\mathbb{Z}, *)$ a group? Explain.

17. (6 points) S be a set and let B be a subset of S. Define

$$H = \{\sigma \in \mathrm{Sym}(S) \mid \sigma(b) \in B \text{ for all } b \in B\}.$$

Suppose $S = \{1, 2, 3, 4, 5, 6\}$ and $B = \{1, 3, 5\}$. How many elements does H have? Explain.