~~60~~ 73

17. How many permutations in S_6 have order 4. Explain your answer.

There are $\binom{6}{4} \cdot 3!$ 4-cycles in S_6

There are $\binom{6}{4} \cdot 3!$ permutations of the shape $(- - - -)(- -)$ in S_6.

So S_6 has $\binom{6}{4} \cdot 3! + \binom{6}{4} \cdot 3!$ permutations of order 4.

$$= 15 \cdot 6 + 15 \cdot 6 = 180$$

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18. Let G be the group of non-zero complex numbers under multiplication. Let G' be the group of non-zero 2×2 matrices with real entries of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ under multiplication. Consider the function $\varphi \colon G \to G'$, which is given by $\varphi(a+bi) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$. Prove that φ is a group isomorphism.

It is obvious that φ is onto. A typical element of G' is $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ with either $a \neq 0$ or $b \neq 0$. We see that $a+bi \in G$ and $\varphi(a+bi) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$. It is also obvious that φ is 1-1. Suppose $a+bi$ and $c+di$ are in G with $\varphi(a+bi) = \varphi(c+di)$. So $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \begin{bmatrix} c & d \\ -d & c \end{bmatrix}$ so $a = c$ & $b = d$. So $a+bi = c+di$.

7

φ resp. mult

$$\varphi(a+bi) \cdot \varphi(c+di) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \begin{bmatrix} ac-bd & ad+bc \\ -ad-bc & ac-bd \end{bmatrix}$$

these are equal

$$\varphi((a+bi)(c+di)) = \varphi((ac-bd)+i(ad+bc)) = \begin{bmatrix} ac-bd & ad+bc \\ -(ad+bc) & ac-bd \end{bmatrix}$$