

5. Let  $a$  and  $b$  be elements of finite order in a group  $G$ . LIST two hypotheses so that if  $a$  and  $b$  satisfy these hypotheses, then the order of  $ab$  is equal to the order of  $a$  times the order of  $b$ . PROVE the result.

Let  $\theta(x)$  mean the ord

Hyp 1  $\theta(a)$  and  $\theta(b)$  are relatively prime

Hyp 2  $a^b = b^a$

Proof Let  $r = \theta(a)$  and  $s = \theta(b)$ . We will show that  $rs = \theta(ab)$

It is clear that  $(ab)^{rs} = (ab)^s (b^a)^r = id^s id^r = id$

So  $\theta(ab) \leq rs$ . Let  $t = \theta(ab)$ .

We know that  $(ab)^t = id$  so  $a^t = b^{-t} \in \langle a \rangle \cap \langle b \rangle$

$\therefore \theta(a^t) \mid \theta(a)$  and  $\theta(b^t) \mid \theta(b)$  (since  $a^t \in \langle a \rangle$  and  $b^t \in \langle b \rangle$ )

But  $\theta(a)$  and  $\theta(b)$  are relatively prime so  $\theta(a^t) = 1$  so  $a^t = id$

so  $r \mid t$

Also  $b^t = (a^t)^{-1} = id$  so  $s \mid t$ . But  $r$  and  $s$  are relatively prime.

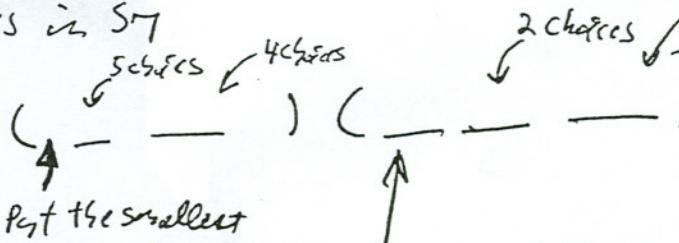
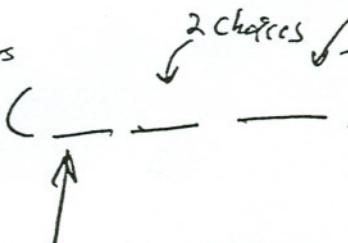
Since  $r \mid t$  and  $s \mid t$ ; thus,  $rs \mid t$

we have  $r, s, t$  all positive with  $t \leq rs$  and  $rs \mid t$ . It follows

in other words  $\theta(a)\theta(b) = \theta(ab)$

6. How many permutations in  $S_7$  have order 3? Explain your answer.

There are  $2 \cdot \binom{7}{3}$  3-cycles in  $S_7$

Then there are  $\binom{7}{6} \cdot 40$  derangements (  ) (  )

The total number of permutations in  $S_7$  of order 3

is

$$2 \cdot \frac{7 \cdot 6 \cdot 5}{3!} + 7 \cdot 40$$

$$= 70 + 280$$

$$= \boxed{350}$$