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5. Let A be a set, B be a subset of A, and b be an element of B. Is

$$G = \{\sigma \in \mathrm{Sym}(A) \mid \sigma(b) \in B\}$$

always a subgroup of S_A? If your answer is yes, then PROVE the statement. If your answer is no, then give a COUNTEREXAMPLE.

G is NOT always a group. For example Take $A = \{1, 2, 3\}$
$B = \{1, 2\}$ and $b = 1$

I see that $\sigma = (1\,2\,3) \in G$ because $\sigma(1) = 2 \in B$
But $\sigma \circ \sigma \notin G$ (so G is not closed) because
$\sigma \circ \sigma = (1\,3\,2)$ and $\sigma \circ \sigma(1) = 3 \notin B$

6. Let H be a subgroup of the finite group G. Let $x \in G$, and let $[x] = \{y \in G \mid xy^{-1} \in H\}$. Prove that H and $[x]$ have the same number of elements.

Notice that $y \in [x] \iff xy^{-1} = h$ for some $h \in H \iff h^{-1}x = y$ for some

I create a function from H to $[x]$ which is One-to-one and onto
namely $f : H \longrightarrow [x]$ is given by $f(h) = hx$. We see that
Note $f(h)$ really is in $[x]$. We see that every element of $[x]$ is
equal to $f(h)$ for some $h \in H$. We see that f is one-to-one
(If h_1 and h_2 are different, then h_1x and h_2x are different.)
We conclude that H and $[x]$ have the same number
of elements.