Math 546, Final Exam, Spring 2010

Write everything on the blank paper provided.

You should KEEP this piece of paper.

If possible: turn the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 100 points. There are **10** problems **on TWO sides**. Each problem is worth 10 points.

Write **coherently** in **complete sentences**.

No Calculators or Cell phones.

1. State the First Isomorphism Theorem. Use complete sentences. Write everything that is necessary, but nothing extra.

2. Define *kernel*. Use complete sentences. Write everything that is necessary for your definition to make sense, but nothing extra.

3. Define *left coset*. Use complete sentences. Write everything that is necessary for your definition to make sense, but nothing extra.

4. Let $\varphi\colon G \to G'$ be a group homomorphism. Suppose that the kernel of φ consists of the identity element of G. Prove that φ is a one-to-one.

5. Define a one-to-one homomorphism from the unit circle group U to the General Linear group $\mathrm{GL}_2(\mathbb{R})$.

6. Let V_4 be the subset $\{\mathrm{id}, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$ of the symmetric group S_4. It is true that V_4 is a normal subgroup of S_4; however, you do not have to prove this. What familiar group is isomorphic to $\frac{S_4}{V_4}$? Explain.

7. Let G be a cyclic group of order n. Prove that G has **exactly one** subgroup for each divisor m of n.

8. Suppose that m and n are relatively prime non-zero integers. Prove that the groups $\frac{\mathbb{Z}}{mn\mathbb{Z}}$ and $\frac{\mathbb{Z}}{m\mathbb{Z}} \times \frac{\mathbb{Z}}{n\mathbb{Z}}$ are isomorphic.

9. Consider $\varphi : \frac{\mathbb{Z}}{2\mathbb{Z}} \to \frac{\mathbb{Z}}{4\mathbb{Z}}$, with $\varphi(n+2\mathbb{Z}) = n+4\mathbb{Z}$. Is φ a group homomorphism? Explain thoroughly.

10. Let $(S, *)$ be the group $(\mathbb{R} \setminus \{-1\}, *)$ with $a * b = ab + a + b$ for all a and b in S . Define the function φ: $(\mathbb{R} \setminus \{0\}, \times) \to (S, *)$ by $\varphi(r) = r - 1$. Prove that φ is a group homomorphism.