Math 546, Final Exam , Fall 2011

Write everything on the blank paper provided.

You should KEEP this piece of paper.

If possible: turn the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam. The exam is worth 100 points. There are **13** problems.

Write **coherently** in **complete sentences**. **No Calculators or Cell phones.**

1. (7 points) Prove that $\frac{\mathbb{Z}}{n\mathbb{Z}} \times \frac{\mathbb{Z}}{n\mathbb{Z}} \to \frac{\mathbb{Z}}{n\mathbb{Z}}$, given by $(a+n\mathbb{Z}, b+n\mathbb{Z}) \mapsto ab+n\mathbb{Z}$, is a well-defined function.

2. (7 points) Recall that $\mathbb{Z}_n^\times$ is the set of cosets $a+n\mathbb{Z}$ in $\frac{\mathbb{Z}}{n\mathbb{Z}}$ where a and n are relatively prime integers. You proved for homework that $\mathbb{Z}_n^\times$ is a group under the operation of problem 1. What is the inverse of $38+105\mathbb{Z}$ in $\mathbb{Z}_{105}^\times$?

3. (7 points) Recall the definition of the group $\mathbb{Z}_{16}^\times$ from problem 2. Is this group cyclic? Explain very thoroughly.

4. (7 points) Is $\frac{\mathbb{Z}}{3\mathbb{Z}} \to \frac{\mathbb{Z}}{6\mathbb{Z}}$, with $a+3\mathbb{Z} \mapsto a+6\mathbb{Z}$ a function? Explain very thoroughly.

5. (8 points) Is $\frac{\mathbb{Z}}{6\mathbb{Z}} \to \frac{\mathbb{Z}}{3\mathbb{Z}}$, with $a+6\mathbb{Z} \mapsto a+3\mathbb{Z}$ a function? Explain very thoroughly.

6. (8 points) Define a group homomorphism from $\mathbb{Z} \times \mathbb{Z}$ onto $\mathbb{Z}$ whose kernel is the subgroup of $\mathbb{Z} \times \mathbb{Z}$ generated by $(1,1)$. Apply the First Isomorphism Theorem.

7. (8 points) Prove that the groups $\frac{\mathbb{R}}{4\mathbb{Z}}$ and U are isomorphic.

8. (8 points) Let G be a cyclic group of order n . **Prove** that G has exactly one subgroup of order m for each divisor m of n .

9. (8 points) **State and prove** the Chinese Remainder Theorem.

10. (8 points) **State** the **best** Theorem we proved concerning the order of a product. Be sure to list all of the hypotheses.

11. (8 points) Give an example of **one** set X and two functions $f\colon X \to X$ and $g\colon X \to X$ such that $(g \circ f)(x) = x$ for all $x \in X$, but f is not onto.
    PLEASE TURN OVER.

12. (8 points) Let G be a group with 35 elements. Suppose that G has exactly one subgroup of order 5 and exactly one subgroup of order 7. Prove that G is a cyclic group.

13. (8 points) Let G be a finite group. Suppose that H is a subgroup of G and the order of H is exactly one half the order of G. Prove that H is a normal subgroup of G.